Choice through a unified lens: The prudential model*

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Abstract

We present a new choice model. An agent is endowed with two sets of preferences: pro-preferences and con-preferences. For each choice set, if an alternative is the best (worst) for a *pro-preference* (*con-preference*), then this is a *pro* (*con*) for choosing that alternative. The alternative with more pros than cons is chosen from each choice set. Each preference may have a *weight* reflecting its salience. In this case, the probability that an alternative is chosen equals the difference between the weights of its pros and cons. We show that this model provides a unified lens through which every nuance of the rich human choice behavior can be structurally explained. *JEL* Classification Numbers: D01, D07, D09.

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1 Introduction

Charles Darwin, the legendary naturalist, wrote "The day of days!" in his journal on November 11, 1838, when his cousin Emma Wedgwood accepted his marriage proposal. However, whether to marry at all had been a hard decision for Darwin. Just a few months prior, Darwin had scribbled a carefully considered list of *pros* –such as "constant companion", "charms of music", "female chit-chat"–and *cons* –such as "may be quarrelling", "fewer conversations with clever people", "no books"– regarding the potential impact of marriage on his life.[1] With this list of pros and cons, Darwin seems to follow a choice procedure ascribed to Benjamin Franklin. Here we present Franklin (1887)'s choice procedure in his own words.

> *To get over this, my Way is, to divide half a Sheet of Paper by a Line into two Columns, writing over the one Pro, and over the other Con. I endeavour to estimate their respective Weights; and where I find two, one on each side, that seem equal, I strike them both out: If I find a Reason pro equal to some two Reasons con, I strike out the three. If I judge some two Reasons con equal to some three Reasons pro, I strike out the five; and thus proceeding I find at length where the Ballance lies. And tho' the Weight of Reasons cannot be taken with the Precision of Algebraic Quantities, yet when each is thus considered separately and comparatively, and the whole lies before me, I think I can judge better, and am less likely to take a rash Step; and in fact I have found great Advantage from this kind of Equation, in what may be called Moral or Prudential Algebra.*

Choice models most commonly used in economics are based on maximization of preferences. An alternative mode of choice, which is common for the scholarly work in other social disciplines such as history, law, and political science, is the less formal *reason-based analysis* (Shafir et al. (1993)). In the vein of Franklin's prudential algebra, first, various arguments that support or oppose an alternative are identified, then the balance of these arguments determines the choice.[2] We formu-

[1] See Glass (1988) for the full list.

[2] Shafir et al. (1993) argue that reason-based analyses have been used to understand unique his-

late and analyze the *prudential choice model* that connects these two approaches by presenting a reason-based choice model, in which the 'reasons' are formed by using a preference-based language.

In the last decade several choice models have been proposed to accommodate choice behavior that classical theories fail to explain. In this study we observe that every nuance of the rich human choice behavior can be captured via a structured model that exhibits limited context-dependency. The random prudential model renders an *additive* and *structurally-invariant* representation, similar to the random utility model. This reflects itself as a form of uniqueness in the representation, which facilitates the model's identification. In our examples, we present specific prudential models that accommodate observed choice behavior that commonly used models fail to explain–such as *similarity effect* and *attraction effect*–by capturing the key aspects of the contexts in which these choice patterns are observed.

First, we formulate the prudential choice model in the deterministic choice setup by extending Franklin's prudential algebra to choice sets that possibly contain more than two alternatives. A *choice function* C singles out an alternative from each *choice set* S, which is a nonempty subset of the grand *alternative set* X. A *(deterministic) prudential model* (PM) is a pair $\langle \succ, \rhd \rangle$ such that $\succ = \{\succ_1, \cdots, \succ_m\}$ is a collection of *pro-preferences* and $\rhd = \{\rhd_1, \cdots, \rhd_q\}$ is a collection of *con-preferences*. Given an PM $\langle \succ, \rhd \rangle$, for each choice set S and alternative x, if x is the $\succ_i$-best alternative in S for some $\succ_i \in \succ$, then we interpret this as a 'pro' for choosing x from S. On the other hand, if x is the $\rhd_i$-worst alternative in S for some $\rhd_i \in \rhd$, then we interpret this as a 'con' for choosing x from S. More formally, $Pros(x, S)$ denotes the set of pro-preferences ($\succ_i \in \succ$) at which x is the best alternative in S and $Cons(x, S)$ denotes the set of con-preferences ($\rhd_i \in \rhd$) at which x is the worst alternative in S. Our central new concept is the following: A choice function is *prudential* if there is an PM $\langle \succ, \rhd \rangle$ such that for each choice set S, an alternative x is chosen from S if and only if

toric, legal and political decisions. Examples include presidential decisions taken during the Cuban missile crisis (Allison (1971)), the Camp David accords (Telhami (1990)), and the Vietnam war (Gelb & Betts (2016)). Reason-based analysis is also commonly used for the analysis of 'case studies' in business and law schools.

the number of $Pros(x, S)$ is greater than the number of $Cons(x, S)$.

Next, we formulate the prudential model in the stochastic choice setup. In this setup, an agent's repeated choices or a group's choices are summarized by a *random choice function* (RCF) p, which assigns to each choice set S, a probability measure over S. For each choice set S and alternative x, we denote by $p(x, S)$ the probability that alternative x is chosen from choice set S. A *random prudential model* (RPM) is a triplet $\langle \succ, \rhd, \lambda \rangle$, where $\succ$ and $\rhd$ stand for pro-preferences and con-preferences, as before. The weight function λ assigns to each pro-preference $\succ_i \in \succ$ and con-preference $\rhd_i \in \rhd$, a value in the $(0, 1]$ interval, which we interpret as a measure of the salience of each preference. In line with the experimental findings of Shafir (1993) indicating that the weight assigned to the pros is more than the weight assigned to the cons, we require that the difference between the weighted sum of pro-preferences and con-preferences is unity. An RCF p is *prudential* if there is an RPM $\langle \succ, \rhd, \lambda \rangle$ such that for each choice set S and alternative x,

$$p(x, S) = \lambda(Pros(x, S)) - \lambda(Cons(x, S)),$$

where $\lambda(Pros(x, S))$ and $\lambda(Cons(x, S))$ are the sum of the weights over $Pros(x, S)$ and $Cons(x, S)$.

The most familiar stochastic choice model in economics is the *random utility model* (RUM), which assumes that an agent is endowed with a probability measure μ over a set of preferences $\succ$ such that he randomly selects a preference to be maximized from $\succ$ according to μ. Each RUM $\langle \succ, \mu \rangle$ is an RPM in which there is no set of con-preferences. As for the similarity between the RPM and the RUM, both models are *additive*, in the sense that the choice probability of an alternative is calculated by summing up the weights assigned to the preferences. The primitives of both the RPM and RUM are structurally invariant, in the sense that the decision maker uses the same $\langle \succ, \mu \rangle$ and $\langle \succ, \rhd, \lambda \rangle$ to make a choice from each choice set. These two features of RUM brings stringency in its identification, which reflects itself in its characterization.[3] On the other hand, despite the similarity between the RPM and the RUM, in

[3]Namely, the RCFs that render a random utility representation are those with nonnegative Block-Marschak polynomials. See Block & Marschak (1960), Falmagne (1978), McFadden (1978), and

our Theorem 1, we show that every random choice function is prudential. Then, by using the construction in Theorem 1's proof–based on a technical extension of Ford Jr & Fulkerson (2015)'s seminal result in optimization theory–and two key results from the integer-programming literature, we show that each (deterministic) choice function is prudential.

Our main results show that the prudential model–an additive model similar to the RUM–provides a canonical language to describe any choice behavior in terms of structurally-invariant primitives. It seems that what makes a choice model economically interesting is twofold. One concern is whether the primitives of the model can be precisely identified from the observed choices. The other concern is whether the model provides plausible explanations for observed choice patterns that classical models fail to explain. In the rest of the paper, we aim to address these concerns.

In Section 2.2, we present specific prudential choice models that accommodate observed choice behavior, such as the *similarity effect* and the *attraction effect* that commonly used choice models fail to explain. This may seem of little importance for an inclusive choice model, however, our point is to illustrate that tailored prudential choice models capture the key aspects of the contexts in which these choice patterns are observed. For example, in the classical attraction effect scenario it seems that there are only two relevant criteria for choice, such as price and quantity. The pro- and con-preferences used in our Example 3 correspond to these criteria. As a result, the choice probability of an alternative may increase when a *decoy* is added, since this alternative may no longer be the worst one according to a relevant criterion. A key feature that derives the similarity effect is that there are two distinct attributes that are relevant for choice, one of which is of major importance, whereas the other is of secondary importance. The pro- and con-preferences used in our Example 2 reflects this logic. As for the identification of the primitives from observed choices, the RPM has characteristics similar to the RUM that we discuss in Section 2.4. Our results facilitate identification of other inclusive choice models, which otherwise may be rather difficult. In Section 3.3, we present an application along these lines, in which we show that each choice function is *plurality-rationalizable*.

Barberá & Pattanaik (1986).

1.1 Related literature

In the deterministic choice literature, previous choice models proposed by Kalai et al. (2002) and Bossert & Sprumont (2013) yield similar "anything goes" results. A choice function is *rationalizable by multiple rationales* (Kalai et al. (2002)) if there is a collection of preference relations such that for each choice set the choice is made by maximizing one of these preferences. A choice function is *backwards-induction rationalizable* (Bossert & Sprumont (2013)) if there is an extensive-form game such that for each choice set the backwards-induction outcome of the restriction of the game to the choice set coincides with the choice. In this model, for each choice set, a new game is obtained by pruning the original tree of all branches leading to unavailable alternatives. In the stochastic choice setup, Manzini & Mariotti (2014) provide an anything-goes result for the *menu-dependent random consideration set rules*. In this model, an agent keeps a single preference relation and attaches to each alternative a choice-set-specific attention parameter. Then, from each choice he chooses an alternative with the probability that no more-preferable alternative grabs his attention. In contrast to these models, we believe that the prudential model is more structured, and exhibits limited context dependency. In that, an agent following a prudential model only restricts the pro-preferences and con-preferences to the given choice set to make a choice.

Our Theorem 1 is partly related to a result in a contemporary paper by Saito (2017), who offers characterizations of the mixed logit model. It follows from Proposition 3 of this paper, which is proved by using a different approach, that each RCF can be expressed as an affine combination of two random utility functions. This result renders a prudential representation without requiring weights be at most unity.[4] Besides the contemporaneous nature and different focus of the two papers, requiring weights be at most unity is critical for our results, and poses additional technical challenges. We highlight these in Remark 1 and Section 2.3.

[4] Saito (2017) reports this observation in footnote 7 on p. 15. We are grateful to an anonymous referee for bringing this connection to our awareness.

2 Prudential random choice functions

2.1 The model

Given a nonempty finite alternative set X, any nonempty subset S is called a **choice set**. Let Ω denote the collection of all choice sets. A **random choice function** (RCF) p is a mapping that assigns each choice set $S \in \Omega$, a probability measure over S. For each $S \in \Omega$ and $x \in S$, we denote by $p(x, S)$ the probability that alternative x is chosen from choice set S. A *preference*, denoted generically by $\succ_i$ or $\rhd_i$, is a complete, transitive, and antisymmetric binary relation on X.

A **random prudential model** (RPM) is a triplet $\langle \succ, \rhd, \lambda \rangle$, where $\succ = \{\succ_1, \cdots, \succ_m\}$ and $\rhd = \{\rhd_1, \cdots, \rhd_q\}$ are sets of pro- and con-preferences on X. We assume that if $\succ_i$ is a pro-preference, then there is no con-preference $\rhd_i$ which is the inverse of $\succ_i$. That is, being best according to a preference should not simultaneously be a pro and con for an alternative. Finally, the **weight function**, denoted by λ is such that for each $\succ_i \in \succ$ and $\rhd_i \in \rhd$, we have $\lambda(\succ_i) \in (0, 1]$, $\lambda(\rhd_i) \in (0, 1]$, and the difference between the weighted sum of pro-preferences and con-preferences is one, i.e. $\sum_{\{\succ_i \in \succ\}} \lambda(\succ_i) - \sum_{\{\rhd_i \in \rhd\}} \lambda(\rhd_i) = 1$. The weight function λ acts like a probability measure over the set of preferences that can assign negative values. We interpret the weight assigned to each pro-preference or con-preference as a measure of the salience of that preference. To define when an RCF is prudential, let $Pros(x, S) = \{\succ_i \in \succ : \ x = max(S, \succ_i)\}$ and $Cons(x, S) = \{\rhd_i \in \rhd \ : \ x = min(S, \rhd_i)\}$.

Definition 1 *An RCF p is* ***prudential*** *if there is an RPM $\langle \succ, \rhd, \lambda \rangle$ such that for each choice set $S \in \Omega$ and $x \in S$,*

$$p(x, S) = \lambda(Pros(x, S)) - \lambda(Cons(x, S)), \tag{D1}$$

where $\lambda(Pros(x, S))$ and $\lambda(Cons(x, S))$ are the sum of the weights over $Pros(x, S)$ and $Cons(x, S)$.

As the reader would easily notice not every RPM $\langle \succ, \rhd, \lambda \rangle$ yields an RCF. For this to be true, for each choice set $S \in \Omega$ and $x \in S$, expression in (D1) should be nonnegative and sum up to one. These additional requirements are imposed on the model by our Definition 1. Next, we provide a less structured formulation of the prudential model that always yields an RCF. For a given RPM $\langle \succ, \triangleright, \lambda \rangle$, let for each $S \in \Omega$ and $x \in S$, $\lambda(x, S) = \lambda(Pros(x, S)) - \lambda(Cons(x, S))$, and $S^+ = \{x \in S : \lambda(x, S) > 0\}$. It directly follows that if an RCF p can be represented as in D1, then p can be represented as in D2. Since Definition 1 is a rather parsimonious one, converse does not follow directly. This parsimony derives the uniqueness result in Proposition 1, and paves the way for obtaining our Theorem 2.

Definition 2 *An RCF p is* ***prudential*** *if there is an RPM $\langle \succ, \triangleright, \lambda \rangle$ such that for each choice set $S \in \Omega$ and $x \in S$,*

$$p(x, S) = \begin{cases} \dfrac{\lambda(x, S)}{\sum_{\{y \in S^+\}} \lambda(y, S)} & \text{if } \lambda(x, S) > 0 \\ 0 & \text{if } \lambda(x, S) \leq 0 \end{cases} \tag{D2}$$

That is, to make a choice from each choice set S, a prudential agent considers the alternatives with a positive $\lambda(x, S)$ score, and chooses each alternative from this consideration set with a probability proportional to its weight.

We can render an intuitive interpretation for the RPM in the vein of Tversky (1972)'s *elimination by aspects*, in which an agent views each alternative as a set of attributes and makes his choice by following a probabilistic process that eliminates alternatives based on their attributes. To see the connection, consider a con-preference $\triangleright_i$; if an alternative x is not the $\triangleright_i$-worst alternative in a choice set S, then say that x is *acceptable* according to $\triangleright_i$ in S. Now, we can interpret the statement "*x has attribute i in choice set S*" as "*x is acceptable according to $\triangleright_i$ in S*". Thus, for a given RPM, each alternative without attribute i in choice set S is eliminated with a probability proportional to the weight of attribute i. In line with this interpretation, we illustrate in our Example 2 and Example 3 that each preference in an RPM can be interpreted as an attribute or a relevant criterion for the choice. The agent's attitude to these criteria is different in that if it is a pro-preference, then he seeks maximization; if it is a con-preference, then he is satisfied by the elimination of the worst alternative.

Remark 1 We require the weights be at most unity for each pro- or con-preference.

As for Definition 1, this is not a simple normalization exercise. To see this, suppose an RCF p is represented as in D1 where some weights are above one. If we divide all the terms with a common element, then D1 no longer holds. Besides technical challenges posed by requiring each weight be at most unity, it is critical for obtaining Theorem 2

2.2 Examples

Following examples present specific prudential choice models that accommodate observed choice behavior that commonly used choice models fail to explain. Our point is to illustrate that the tailored prudential choice models capture the key aspects of the contexts in which these choice patterns are observed. the RPM.

Example 1 (Binary choice cycles) Suppose $X = \{x, y, z\}$ and consider the following RPM $\langle \succ, \rhd, \lambda \rangle$. Note that x is chosen from the grand set and when compared to

$$\begin{array}{cc|c} (1) & (1) & (1) \\ \succ_1 & \succ_2 & \rhd_1 \\ \hline x & y & z \\ y & z & x \\ z & x & y \end{array}$$

y, y is chosen when compared to z, but z is chosen when compared to x. That is, the given PM generates the choice behavior of an agent who exhibits a binary choice cycle between x, y, z, and chooses x from the grand set.

Example 2 (Similarity Effect) Suppose $X = \{x_1, x_2, y\}$, where x_1 and x_2 are similar alternatives, such as recordings of the same Beethoven symphony by different conductors, while y is a distinct alternative, such as a Debussy suite. Suppose between any pair of the three recordings our classical music aficionado chooses with equal probabilities, and he chooses from the set $\{x_1, x_2, y\}$ with probabilities $0.25, 0.25$, and 0.5 respectively.[5] Consider the RPM $\langle \succ, \rhd, \lambda \rangle$ presented below:

[5] Debreu (1960) proposes this example to highlight a shortcoming of the Luce rule (Luce (1959)).

(1/4)	(1/4)	(1/2)	(1/2)
$\succ_1$ / $\rhd_1$	$\succ_2$ / $\rhd_2$	$\succ_3$	$\succ_4$
y	y	x_1	x_2
x_1	x_2	x_2	x_1
x_2	x_1	y	y

We choose $(\succ_1, \rhd_1)$ and $(\succ_2, \rhd_2)$ as the same preferences, and assign the same weight. In the story, the composer has primary importance, whereas the conductor has secondary importance. In line with this observation, all the preferences in the given RPM ranks the recordings first according to composer, then according to conductor. One can easily verify that the induced RCF generates our classical music aficionado's choices.

In Example 2, there are two alternatives that are slightly different. If the substitution is not extreme, then an agent may exhibit a choice pattern incompatible with the RUM. In this vein, the next example illustrates that when we introduce an asymmetrically dominated alternative, the choice probability of the dominating alternative may go up. This choice behavior, known as the *attraction effect*, is incompatible with any RUM.[6]

Example 3 (Attraction Effect) Suppose $X = \{x, y, z\}$, where x and y are two competing alternatives such that none clearly dominates the other, and z is another alternative that is dominated by x but not y. To illustrate the attraction effect, we follow the formulation in our Definition D2. Consider the following RPM $\langle \succ, \rhd, \lambda \rangle$, in which there is single pair of preferences used both as the pro- and con-preferences. We can interpret this preference pair as two distinct criteria that order the alternatives.

Now, since for both criteria x is better than z, we get $p(x, \{x, z\}) = 1$. Since x

This phenomena is later referred to as the *similarity effect* or *duplicates effect*. See Gul et al. (2014) for a random choice model that accommodates the similarity effect.

[6]Experimental evidence for the attraction effect is first presented by Payne & Puto (1982) and Huber & Puto (1983). Following their work, evidence for the attraction effect has been observed in a wide variety of settings. For a list of these results, consult Rieskamp et al. (2006).

(1/2)	(1/2)	(1/4)	(1/4)
$\succ_1$	$\succ_2$	$\rhd_1$	$\rhd_2$
x	y	x	y
z	x	z	x
y	z	y	z

and y fail to dominate each other, and y fail to dominate z, we get $p(y, \{x, y\}) = p(y, \{y, z\}) = 1/2$. That is, z is a 'decoy' for x when y is available. Note that when only x and y are available, since x is the $\rhd_2$-worst alternative, x is eliminated with a weight of $1/2$,. However, when the decoy z is added to the choice set, then x is no longer the $\rhd_2$-worst alternative, and we get $p(x, \{x, y, z\}) = 2/3$. That is, availability of decoy z increases the choice probability of x. Thus, the proposed RPM presents an attraction effect scenario.

2.3 Main result

In our main result, we show that every random choice function is prudential. We present this rather technical proof in Section A of the online appendix. Next, we state the theorem and discuss the connection to Saito (2017).

Theorem 1 *Every random choice function is prudential.*

As discussed in Section 1.1, it follows from the results of a contemporary paper by Saito (2017) that each RCF can be expressed as an affine combination of two random utility functions. It follows from our Theorem 1 that the weights used in this affine combination can be chosen from $[-1, 1]$ interval. To see the technical difference, note that by following the construction in our proof and directly applying the the Ford-Fulkerson Theorem, without using several results that we obtain, each RCF can be expressed as an affine combination of random utility functions. On the other hand, to show that these weights can be chosen from $[-1, 1]$ interval, we extend the Ford-Fulkerson Theorem (see Lemma 3 in the online appendix) and follow a deliberate induction argument supported by other structural results, such as Lemma

5 in the online appendix.

2.4 Uniqueness

The primitives of the RUM model are structurally invariant in the sense that the agent uses the same $\succ$ and μ to make a choice from each choice set. This feature of the RUM brings precision in identifying the choice behavior. To elaborate on this, although an RCF may have different random utility representations even with disjoint sets of preferences, Falmagne (1978) argues that random utility representation is essentially unique. That is, the sum of the probabilities assigned to the preferences at which an alternative x is the k^{th}-best in a choice set S is the same for all random utility representations of the given RCF. Similarly, the primitives of an RPM are structurally invariant in the sense that the agent uses the same triplet $\langle \succ, \triangleright, \lambda \rangle$ to make a choice from each choice set. As a particular instance of this similarity, both models render a unique representation when there are only three alternatives.[7] As for the general case, our Proposition 1 provides a uniqueness result for the RPM, which can be thought as the counterpart of Falmagne's result for the RUM.

For a given RPM $\langle \succ, \triangleright, \lambda \rangle$, let for each $S \in \Omega$ and $x \in S$, $\lambda(x = B_k|S, \succ)$ be the sum of the weights assigned to the pro-preferences at which x is the k^{th}-best alternative in S. Similarly, let $\lambda(x = W_k|S, \triangleright)$ be the sum of the weights assigned to the con-preferences at which x is the k^{th}-worst alternative in S.

Proposition 1 *If $\langle \succ, \triangleright, \lambda \rangle$ and $\langle \succ', \triangleright', \lambda' \rangle$ are random prudential representations of the same RCF p, then for each $S \in \Omega$ and $x \in S$,*

$$\lambda(x = B_k|S, \succ) - \lambda(x = W_k|S, \triangleright) = \lambda'(x = B_k|S, \succ') - \lambda'(x = W_k|S, \triangleright').$$

This result tells that for each RCF the difference between the the sum of the weights assigned to the pro-preferences at which x is the k^{th}-best alternative in S and the sum of the weights assigned to the con-preferences at which x is the k^{th}-worst alternative in S is the same for each prudential representation of the given RCF. That

[7]This directly follows from the construction used to establish the base of induction in Theorem 1's proof.

is, $\lambda(x = B_k|S, \succ) - \lambda(x = W_k|S, \rhd)$ is fixed for each RPM $\langle \succ, \rhd, \lambda \rangle$ that represents the given RCF. The proof of Proposition 1 follows from the specific construction used to prove Theorem 1. For the proof see Section B in the online appendix.

3 Prudential deterministic choice functions

3.1 The model

A (deterministic) choice function C is a mapping that assigns each choice set $S \in \Omega$ a member of S, that is $C : \Omega \to X$ such that $C(S) \in S$. Let $\succ$ and $\rhd$ stand for two collections of preferences on X as before. A **(deterministic) prudential model (PM)** is a pair $\langle \succ, \rhd \rangle$ consisting of the pro-preferences and the con-preferences. As before, define $Pros(x, S) = \{\succ_i \in \succ : \ x = max(S, \succ_i)\}$ and $Cons(x, S) = \{\rhd_i \in \rhd : x = min(S, \rhd_i)\}$.

Definition 3 *A choice function C is **prudential** if there is an PM $\langle \succ, \rhd \rangle$ such that for each choice set $S \in \Omega$ and $x \in S$, $C(S) = x$ if and only if $|Pros(x, S)| > |Cons(x, S)|$.*

Note that our prudential model is not a direct adaptation of its random counterpart. In that, we require each preference to have a fixed unit weight, instead of having fractional weights. Moreover, if an agent is prudential, then at each choice set S there should be a single alternative x such that the number of $Pros(x, S)$ is greater than the number of $Cons(x, S)$. Therefore, each prudential model $\langle \succ, \rhd \rangle$ may not render a well-defined choice function. However, as in the random setup, one can consider a less structured formulation that always renders a choice function. For this alternative formulation, suppose that for each given prudential model $\langle \succ, \rhd \rangle$, and choice set S, first, the alternatives with the maximum $Pros(x, S) - Cons(x, S)$ value are shortlisted, then one of them is singled out. It directly follows that if a choice function C renders a prudential representation as specified in Definition 3, then C can also be represented in this alternative form. However, showing the converse is a nontrivial exercise.

3.2 Main result

We show that every choice function is prudential. This result does not directly follow from Theorem 1, since a prudential model is not a direct adaptation of the random prudential model. In that we require each preference to have a fixed unit weight instead of having fractional weights. We prove Theorem 2 by using the construction in the proof of Theorem 1 and two well-known results from integer-programming literature. To best of our knowledge the use of integer programming techniques in this context is new. Next, we present the result and its proof.

Theorem 2 *Every choice function is prudential.*

Proof. We prove this result by following the construction used to prove Theorem 1 in the online appendix. So, we proceed by induction. The structure of this induction argument is similar to the one followed by Falmagne (1978) and Barberá & Pattanaik (1986) for the characterization of RUM. Note that since C is a deterministic choice function, for each $x_i \in X$, $\lambda^1([\succ^{x_i}]) \in \{0, 1\}$. Next, by proceeding inductively, we assume that for any $k \in \{1, \ldots, n-1\}$, there is a signed weight function λ^k that takes values $\{-1, 0, 1\}$ over $\mathcal{P}^k$ and represents C_k. It remains to show that we can construct λ^{k+1} taking values $\{-1, 0, 1\}$ over $\mathcal{P}^{k+1}$, and that represents C_{k+1}. We know from Step 1 of the proof of Theorem 1 that to show this it is sufficient to construct λ^{k+1} such that (RS) and (CS) holds. However, this time, in addition to satisfying (RS) and (CS), we require each $\lambda_{ij}^{k+1} \in \{-1, 0, 1\}$.

First, note that equalities (RS) and (CS) can be written as a system of linear equations: $A\lambda = b$, where $A = [a_{ij}]$ is a $(k! + (n-k)) \times (n-k)k!$ matrix with entries $a_{ij} \in \{0, 1\}$, and $b = [\lambda^k([\succ_1^k]), \ldots, \lambda^k([\succ_{k!}^k]), q(x_1, S), \ldots, q(x_{n-k}, S)]$ is the column vector of size $k! + (n-k)$. Let Q denote the associated polyhedron, i.e. $Q = \{\lambda \in \mathbb{R}^{(n-k)k!} : A\lambda = b \text{ and } -1 \leq \lambda \leq 1\}$. A matrix is **totally unimodular** if the determinant of each square submatrix is 0, 1 or -1. It directly directly follows from Theorem 2 of Hoffman & Kruskal (2010) that if the matrix A is totally unimodular, then the vertices of Q are integer valued.

Heller & Tompkins (1956) provide a set of sufficient conditions for a matrix being totally unimodular. See Section C in the online appendix for this result. Next, we show that the matrix that is used to define (RS) and (CS) as a system of linear equations is totally unimodular, by verifying this set of sufficient conditions. To see this, let A be the matrix defining the polyhedron Q. Since $A = [a_{ij}]$ is a matrix with entries $a_{ij} \in \{0, 1\}$, (1) and (4) are directly satisfied. To see that (2) and (3) also hold, let $R_1 = [1, \ldots, k!]$ consist of the the first $k!$ rows and $R_2 = [1, \ldots, n-k]$ consist of the the remaining $n-k$ rows of A. Note that for each $i \in R_1$, the i^{th} row A_i is such that $A_i\lambda = \lambda^k([\succ_i^k])$. That is, for each $j \in \{(i-1)k!, \ldots, ik!\}$, $a_{ij} = 1$ and the rest of A_i equals 0. For each $i \in R_2$, the i^{th} row A_i is such that $A_i\lambda = q(x_i, A)$. That is, for each $j \in \{i, i+k!, \ldots, i+(n-k-1)k!\}$, $a_{ij} = 1$ and the rest of A_i equals 0. To see that (2) and (3) hold, note that for each $i, i' \in R_1$ and $i, i' \in R_2$, the non-zero entries of A_i and $A_{i'}$ are disjoint. It follows that for each column there can be at most two rows with value 1, one in R_1 and the other in R_2.

Finally, it follows from the construction in Step 3 of the proof of Theorem 1 that Q is nonempty, since there is λ vector with entries taking values in the $[-1, 1]$ interval. Since, as shown above, A is totally unimodular, it follows that the vertices of Q are integer valued. Therefore, λ^{k+1} can be constructed such that (RS) and (CS) hold, and each $\lambda_{ij}^{k+1} \in \{-1, 0, 1\}$. ■

3.3 Plurality-rationalizable choice functions

We propose a collective decision making model based on plurality voting. It turns out that this model is closely related to our prudential choice model. To introduce this model, let $\succ^* = [\succ_1^*, \ldots, \succ_m^*]$ be a preference profile, which is a list of preferences. In contrast to a collection of preferences, denoted by $\succ$, a preference $\succ_i$ can appear more than once in a preference profile $\succ^*$. For each choice set $S \in \Omega$ and $x \in S$, x is the **plurality winner of $\succ^*$ in** S if for each $y \in S \setminus \{x\}$, the number of preferences in $\succ^*$ that top ranks x in S is more than the number of preferences in $\succ^*$ that top ranks y in S. That is, for each $y \in S \setminus \{x\}$, $|\{\succ_i^* : \ x = max(S, \succ_i^*)\}| > |\{\succ_i^* : \ y = max(S, \succ_i^*)\}|$. Next, we define plurality-rationalizability, then by using our Theorem 2, we show that every choice function is plurality-rationalizable.

Definition 4 *A choice function C is* ***plurality-rationalizable*** *if there is preference profile $\succ^*$ such that for each choice set $S \in \Omega$ and $x \in S$, $C(S) = x$ if and only if x is the plurality winner of $\succ^*$ in S.*

Proposition 2 *Every choice function is plurality-rationalizable.*

Proof. Let C be a choice function. It follows from Theorem 2 that C is prudential. Let the PM $\langle \succ, \rhd \rangle$ be such that for each choice set $S \in \Omega$ and $x \in S$, $C(S) = x$ if and only if $|Pros(x, S)| > |Cons(x, S)|$. Now, to construct the desired preference profile, first consider the list of all preferences defined on X. Then, eliminate any preference that belongs to $\rhd$ and add any preference that belongs to $\succ$. Let $\succ^*$ be the obtained preference profile. Next, consider a choice set $S \in \Omega$ and suppose $C(S) = x$. In what follows we show that x is the plurality winner of $\succ^*$ in S. We know that $|Pros(x, S)| > |Cons(x, S)|$ and for each $y \in S \setminus \{x\}$, $|Pros(y, S)| \leq |Cons(y, S)|$. It follows that for each $y \in S \setminus \{x\}$, $|Pros(x, S)| - |Cons(x, S)| > |Pros(y, S)| - |Cons(y, S)|$. Now, note that by construction of $\succ^*$, for each $y \in S$ the number of preferences in $\succ^*$ that top ranks y in S equals the number of all preferences that top ranks y in S added to $|Pros(y, S)| - |Cons(y, S)|$. Since for each $y \in S$, the number of all preferences that top ranks y in S is fixed, it follows that x is the plurality winner of $\succ^*$ in S. ■

In an early paper McGarvey (1953) shows that for each asymmetric and complete binary relation, there exists a preference profile such that the given binary relation is obtained from the preference profile by comparing each pair of alternatives via majority voting. For antisymmetric and complete binary relations (without indifferences), we obtain McGarvey's result, as a corollary to Proposition 2. To see this, note that if we restrict a choice function to binary choice sets, then we obtain an antisymmetric and complete binary relation. Since for binary choices, being a plurality winner means being a majority winner, McGarvey's result directly follows.

4 Conclusion

As we have shown, prudential model provides a unified lens through which any deterministic or stochastic choice behavior can be explained. The structural invariance of the prudential model reflects itself as a form of uniqueness, which facilitates identifying the model's primitives from observed choices. We hope that the analytic results that we obtain here would stimulate some empirical work. In our examples we present specific prudential models that accommodate commonly observed choice behavior by capturing the key aspects of the contexts in which these choice patterns are observed. These examples indicate that analyzing prudential choice model with restricted pro- and con-preferences may lead to insightful results.

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Online Appendix to Choice through a unified lens: The prudential model

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Contents

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A Proof of Theorem 1

A.1 Overview of the proof

First, we present an overview of the proof. For a given RCF p, we show that there is a *signed weight function* λ, which assigns each preference $\succ_i$, a value $\lambda(\succ_i) \in [-1, 1]$ such that λ *represents* p. That is, for each choice set S and $x \in S$, $p(x, S)$ is the sum of the weights over preferences at which x is the top-ranked alternative. Once we obtain this signed weight function λ, let $\succ$ be the collection of preferences that receive positive weights, and $\triangleright$ be the collection of the inverses of the preferences that receive negative weights. Let λ^* be the weight function obtained from λ by assigning the absolute value of the weights assigned by λ. It directly follows that p is prudential with respect to the RPM $\langle \succ, \triangleright, \lambda^* \rangle$. Therefore, to prove the theorem, it is sufficient to show that there exists a signed weight function that represents p. We prove this by induction.

To clarify the induction argument, for $k = 1$, let $\Omega_1 = \{X\}$ and let $\mathcal{P}^1$ consists of n-many equivalence classes such that each class contains all the preferences that top rank the same alternative, irrespective of whether these are chosen with positive probability. That is, for $X = \{x_1, \ldots, x_n\}$, we have $\mathcal{P}^1 = \{[\succ^{x_1}], \cdots, [\succ^{x_n}]\}$, where for each $i \in \{1, \ldots, n\}$ and preference $\succ_i \in [\succ^{x_i}]$, $\max(X, \succ_i) = x_i$. Now for each $x_i \in X$, define $\lambda^1([\succ^{x_i}]) = p(x_i, X)$. It directly follows that λ^1 is a signed weight function over $\mathcal{P}^1$ that represents the restriction of the given RCF to Ω_1, denoted by p_1. By proceeding inductively, it remains to show that we can construct λ^{k+1} over $\mathcal{P}^{k+1}$ that represents p_{k+1}. In Step 1 of the proof we show that finding such a λ^{k+1} boils down to finding a solution to the system of equalities described by *row sums* (RS) and *column sums* (CS).[8] To get an intuition for (RS), while moving from the k^{th}-step to the $(k+1)^{th}$-step, each $[\succ^k]$ is decomposed into a collection $\{[\succ_j^{k+1}]\}_{j \in J}$ such that for each $[\succ_j^{k+1}]$ there exists an alternative x_j that is not linearly ordered by $[\succ^k]$, but placed at $[\succ_j^{k+1}]$ right on top of the alternatives that are not linearly ordered

[8] Up to this point the proof structure is similar to the one followed by Falmagne (1978) and Barberá & Pattanaik (1986) for the charaterization of RUM.

by $[\succ^k]$. Therefore, the sum of the weights assigned to $\{[\succ_j^{k+1}]\}_{j\in J}$ should be equal to the weight assigned to $[\succ^k]$. This gives us the set of equalities formulated in (RS). To get an intuition for (CS), let S be the set of alternatives that are not linearly ordered by $[\succ^k]$. Now, we should design λ^{k+1} such that for each $x_j \in S$, $p(x_j, S)$ should be equal to the sum of the weights assigned to preferences at which x_j is the top-ranked alternative in S. The set of equalities formulated in (CS) guarantees this.[9]

Next, we observe that finding a solution to the system described by (RS) and (CS) can be translated to the following basic problem: Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be two real-valued vectors such that the sum of R equals to the sum of C. Now, for which R and C can we find an $m \times n$ matrix $A = [a_{ij}]$ such that A has row sum vector R and column sum vector C, and each entry $a_{ij} \in [-1, 1]$? Ford Jr & Fulkerson (2015) provide a full answer to this question when R and C are positive real valued.[10] However, a peculiarity of our problem is that the corresponding row and column values can be negative. Indeed, we get nonnegative-valued rows and columns only if the Block-Marschak polynomials are nonnegative, that is, the given p is a RUM. In our Lemma 3, we provide an extension of Ford Jr & Fulkerson (2015)'s result that paves the way for our proof.[11] Then, in Step 2 we show that (RS) equals (CS). In Step 3, by using a structural result presented in Lemma 5, we show that the row and column vectors associated with (RS) and (CS) satisfy the premises of our Lemma 3. This completes the construction of the desired signed weight function.

A.2 The proof

We start by proving some lemmas that are critical for proving the theorem. First, we use a result by Ford Jr & Fulkerson (2015)[12] as Lemma 1. Then, our Lemma 2 follows

[9] A related key observation is our Lemma 4, which we obtain by using the *Mobius inversion*.

[10] Brualdi & Ryser (1991) provides a detailed account of similar results.

[11] Roughly speaking, for extending the result to real-valued vectors, the sum of the absolute values of the rows and columns should respect a specific bound.

[12] This result, as stated in Lemma 1, but with integrality assumptions on R, C, and A follows from Theorem 1.4.2 in Brualdi & Ryser (1991), and they report that Ford Jr & Fulkerson (2015) proves, by using network flow techniques, that the theorem remains true if the integrality assumptions are dropped and the conclusion asserts the existence of a real nonnegative matrix.

directly. Next, by using Lemma 2, we prove Lemma 3, which shows that, under suitable conditions, Lemma 1 holds for any real-valued row and column vectors.

Lemma 1 (Ford Jr & Fulkerson (2015)) *Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be positive real-valued vectors with $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$. There is an $m \times n$ matrix $A = [a_{ij}]$ such that A has row sum vector R and column sum vector C, and each entry $a_{ij} \in [0, 1]$ if and only if for each $I \subset \{1, 2, \ldots, m\}$ and $J \subset \{1, 2, \ldots, n\}$,*

$$|I||J| \geq \sum_{i \in I} r_i - \sum_{j \notin J} c_j. \tag{FF}$$

Lemma 2 *Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be positive real-valued vectors with $0 \leq r_i \leq 1$ and $0 \leq c_j \leq m$ such that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_i$. Then there is an $m \times n$ matrix $A = [a_{ij}]$ such that A has row sum vector R and column sum vector C, and each entry $a_{ij} \in [0, 1]$.*

Proof. Given such R and C, since for each $i \in \{1, 2, \ldots, m\}$, $0 \leq r_i \leq 1$, we have for each $I \subset \{1, 2, \ldots, m\}$, $\sum_{i \in I} r_i \leq |I|$. Then, it directly follows that (FF) holds. ■

Next by using Lemma 2, we prove Lemma 3, which plays a key role in proving Theorem 1.

Lemma 3 *Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be real-valued vectors with $-1 \leq r_i \leq 1$ and $-m \leq c_j \leq m$ such that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$. If $2m \geq \sum_{i=1}^{m} |r_i| + \sum_{j=1}^{n} |c_j|$, then there is an $m \times n$ matrix $A = [a_{ij}]$ such that:*

i. A has row sum vector R and column sum vector C,

ii. each entry $a_{ij} \in [-1, 1]$, and

iii. for each $j \in \{1, \ldots, n\}$, $\sum_{i=1}^{m} |a_{ij}| \leq |c_j| + \max\{0, \frac{\sum_{i=1}^{m} |r_i| - \sum_{j=1}^{n} |c_j|}{n}\}$.

Proof. Since r_i and c_j values can be positive or negative, although the sum of the rows equals the sum of the column, their absolute values may not be the same. We analyze two cases separately, where $\sum_{i=1}^{m} |r_i| \geq \sum_{j=1}^{n} |c_j|$ and $\sum_{i=1}^{m} |r_i| < \sum_{j=1}^{n} |c_j|$. Before proceeding with these cases, first we introduce some notation and make some elementary observations.

For each real number x, let $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$. Note that for each x, $x^+ + x^- = x$. Let $R^+ = [r_1^+, \ldots, r_m^+]$ and $R^- = [r_1^-, \ldots, r_m^-]$. Define the n-vectors C^+ and C^- respectively. Next, let $\Sigma_{R^+} = \sum_{i=1}^m r_i^+$, $\Sigma_{R^-} = \sum_{i=1}^m r_i^-$, $\Sigma_{C^+} = \sum_{j=1}^n c_j^+$ and $\Sigma_{C^-} = \sum_{j=1}^n c_j^-$. That is, $\Sigma_{R^+}(\Sigma_{R^-})$ and $\Sigma_{C^+}(\Sigma_{C^-})$ are the sum of the positive (negative) rows in R and columns in C. Since the sum of the rows equals the sum of the columns, we have $\Sigma_{R^+} + \Sigma_{R^-} = \Sigma_{C^+} + \Sigma_{C^-}$.

For each row vector R and column vector C, suppose for each $i \in \{1, \ldots, m_1\}$, $r_i \geq 0$ and for each $i \in \{m_1 + 1, \ldots, m\}$, $r_i < 0$. Similarly, suppose for each $j \in \{1, \ldots, n_1\}$, $c_j \geq 0$ and for each $j \in \{n_1 + 1, \ldots, n\}$, $c_j < 0$. Now, let $R^1(R^2)$ be the m_1-vector ($(m - m_1)$-vector), consisting of the non-negative (negative) components of R. Similarly, for each column vector C, let $C^1(C^2)$ be the n_1-vector ($(n - n_1)$-vector), consisting of the non-negative (negative) components of C. It directly follows from the definitions that $\sum_{i=1}^{m_1} r_i = \sum_{i=1}^m r_i^+$ and $\sum_{i=m_1+1}^m r_i = \sum_{i=1}^m r_i^-$. Similarly, $\sum_{j=1}^{n_1} c_j = \sum_{j=1}^n c_j^+$ and $\sum_{j=n_1+1}^n c_j = \sum_{j=1}^n c_j^-$.

Case 1: Suppose that $\sum_{i=I} |r_i| \geq \sum_{j\in J} |c_j|$. First, for each $j \in \{1, \ldots, n\}$, let

$$\epsilon_j = \frac{\Sigma_{R^+} - \Sigma_{C^+}}{n}.$$

Note that since $\sum_{i=1}^m |r_i| \geq \sum_{j=1}^n |c_j|$, we have $\Sigma_{R^+} \geq \Sigma_{C^+}$ and $\Sigma_{R^-} \leq \Sigma_{C^-}$. Moreover, since the sum of the rows equals the sum of the columns, we have $\Sigma_{R^+} - \Sigma_{C^+} = \Sigma_{C^-} - \Sigma_{R^-}$. Therefore, by the choice of ϵ_j, we get

$$\sum_{i=1}^m r_i^+ = \sum_{j=1}^n c_j^+ + \epsilon_j \text{ and } \sum_{i=1}^m r_i^- = \sum_{j=1}^n c_j^- - \epsilon_j. \tag{1}$$

Next, consider row-column vector pairs $(R^1, C^+ + \epsilon)$ and $(-R^2, -(C^- - \epsilon))$, where ϵ is the non-negative n-vector such that each ϵ_j is as defined above. It follows from (1) that for both pairs the sum of the rows equals the sum of the columns. Now we apply Lemma 2 to the row-column vector pairs $(R^1, C^+ + \epsilon)$ and $(-R^2, -(C^- - \epsilon))$. It directly follows that there exists a positive $m_1 \times n$ matrix A^+ and a negative $(m - m_1) \times n$ matrix A^- that satisfy (i) and (ii). We will obtain the desired matrix A by augmenting

A^+ and A^-. We illustrate A^+ and A^- below.

$$
\begin{array}{c|ccccc|c}
 & (c_1^+ + \epsilon_1) & (c_2^+ + \epsilon_2) & (c_3^+ + \epsilon_3) & \cdots & (c_n^+ + \epsilon_n) & \\
\hline
r_1 \geq 0 & & & & & & \\
r_2 \geq 0 & & & A^+ & & & \\
\vdots & & & & & & \\
r_{m_1} \geq 0 & & & & & & \\
\hline\hline
 & & & & & & r_{m_1+1} < 0 \\
 & & & A^- & & & \vdots \\
 & & & & & & r_m < 0 \\
\hline
 & (c_1^- - \epsilon_1) & (c_2^- - \epsilon_2) & (c_3^- - \epsilon_3) & \cdots & (c_n^- - \epsilon_n) &
\end{array}
$$

Since A^+ and A^- satisfy (i) and (ii), A satisfies (i) and (ii). To see that A satisfies (iii), for each $j \in \{1, \ldots, n\}$, consider $\sum_{i=1}^m |a_{ij}|$. Note that, by the construction of A^+ and A^-, for each $j \in \{1, \ldots, n\}$,

$$\sum_{i=1}^m |a_{ij}| = c_j^+ + \epsilon_j + (-c_j^- + \epsilon_j) = |c_j| + 2\epsilon_j = |c_j| + 2\frac{\Sigma_{R^+} - \Sigma_{C^+}}{n}. \tag{2}$$

Since for each $j \in \{1, \ldots, n\}$, $c_j = c_j^+ + c_j^-$ such that either $c^+ = 0$ or $c_j^- = 0$, we get $|c_j| = c_j^+ - c_j^-$. To see that (iii) holds, observe that $\sum_{i=1}^m |r_i| - \sum_{j=1}^n |c_j| = \Sigma_{R^+} - \Sigma_{C^+} + \Sigma_{C^-} - \Sigma_{R^-}$. Since the sum of the rows equals the sum of the columns, i.e. $\Sigma_{R^+} + \Sigma_{R^-} = \Sigma_{C^+} + \Sigma_{C^-}$, we also have $\Sigma_{R^+} - \Sigma_{C^+} = \Sigma_{C^-} - \Sigma_{R^-}$. This observation, together with (2), implies that (iii) holds.

Case 2 Suppose that $\sum_{i=1}^m |r_i| < \sum_{j=1}^n |c_j|$. First, we show that there exists a non-negative m-vector ϵ such that

(E1) for each $i \in \{1, \ldots, m\}$, $r_i^+ + \epsilon_i \leq 1$ and $r_i^- - \epsilon_i \geq -1$, and

(E2) $\sum_{i=1}^m r_i^+ + \epsilon_i = \sum_{j=1}^n c_j^+$ (equivalently $\sum_{i=1}^m r_i^- - \epsilon_i = \sum_{j=1}^n c_j^-$) holds.

Step 1: We show that if $\Sigma_{C^+} - \Sigma_{R^+} \leq m - \sum_{i=1}^m |r_i|$, then there exists a non-negative m-vector ϵ that satisfies (E1) and (E2). To see this, first note that $m - \sum_{i=1}^m |r_i| = \sum_{i=1}^m (1 - |r_i|)$. Next, note that, by simply rearranging the terms, we can rewrite (E2) as follows:

$$\sum_{i=1}^m \epsilon_i = \Sigma_{C^+} - \Sigma_{R^+}\,. \tag{3}$$

Since $\Sigma_{C^+} - \Sigma_{R^+} \leq \sum_{i=1}^{m}(1 - |r_i|)$, for each $i \in \{1, \ldots, m\}$, we can choose an ϵ_i such that $0 \leq \epsilon_i \leq 1 - |r_i|$ and (3) holds. It directly follows that the associated ϵ vector satisfies (E1) and (E2).

Step 2: We show that since $2m \geq \sum_{i=1}^{m} |r_i| + \sum_{j=1}^{n} |c_j|$, we have $\Sigma_{C^+} - \Sigma_{R^+} \leq m - \sum_{i=1}^{m} |r_i|$. First, it directly follows from the definitions that

$$\sum_{i=1}^{m} |r_i| + \sum_{j=1}^{n} |c_j| = \Sigma_{R^+} - \Sigma_{R^-} + \Sigma_{C^+} - \Sigma_{C^-} \,.$$

Since the sum of the rows equals the sum of the columns, i.e. $\Sigma_{R^+} + \Sigma_{R^-} = \Sigma_{C^+} + \Sigma_{C^-}$, we also have $\Sigma_{R^+} - \Sigma_{C^-} = \Sigma_{C^+} - \Sigma_{R^-}$. It follows that

$$\Sigma_{C^+} - \Sigma_{R^-} \leq m.$$

Finally, if we subtract $\sum_{i=1}^{m} |r_i|$ from both sides of this equality, we obtain $\Sigma_{C^+} - \Sigma_{R^+} \leq m - \sum_{i=1}^{m} |r_i|$, as desired.

It follows from Step 1 and Step 2 that there exists a non-negative m-vector ϵ that satisfies (E1) and (E2). Now, consider the row-column vector pairs $(R^+ + \epsilon, C^1)$ and $(-(R^- - \epsilon), -C^2)$. Since ϵ satisfies (E1) for each $i \in \{1, \ldots, m\}$, $r_i^+ + \epsilon_i \in [0, 1]$ and $r_i^- - \epsilon_i \in [-1, 0]$. Since ϵ satisfies (E2), for both of the row-column vector pairs the sum of the rows equals the sum of the columns. Therefore, we can apply Lemma 2 to row-column vector pairs $(R^+ + \epsilon, C^1)$ and $(-(R^- - \epsilon), -C^2)$. It directly follows that there exists a positive $m \times n_1$ matrix A^+ and a negative $m \times (n - n_1)$ matrix A^- that satisfy (i) and (ii). We obtain the desired matrix A by augmenting A^+ and A^-. We illustrate A^+ and A^- below.

$$\begin{array}{c|c||c|c}
 & c_1 \quad c_2 \quad \cdots \quad c_{n_1} \geq 0 & & \\
\hline
(r_1^+ + \epsilon_1) & & & (r_1^- - \epsilon_1) \\
(r_2^+ + \epsilon_2) & & & (r_2^- - \epsilon_2) \\
\vdots & \Large A^+ & \Large A^- & \vdots \\
\vdots & & & \vdots \\
(r_m^+ + \epsilon_m) & & & (r_m^- - \epsilon_m) \\
\hline
 & & c_{n_1+1} < 0 \quad \cdots \quad c_n &
\end{array}$$

Since A^+ and A^- satisfy (i) and (ii), A satisfies (i) and (ii). In this case, since we did not add anything to the columns and each entry in $A^+(A^-)$ is non-negative (negative), for each $j \in \{1, \ldots, n\}$, $\sum_{i=1}^{m} |a_{ij}| = |c_j|$. Therefore, A also satisfies (iii).
■

To prove Theorem 1, let p be an RCF and $\mathcal{P}$ denote the collection of all preferences on X. First, we show that there is a **signed weight function** $\lambda : \mathcal{P} \rightarrow [-1, 1]$ that **represents** p, i.e. for each $S \in \Omega$ and $x \in S$, $p(x, S)$ is the sum of the weights over $\{\succ_i \in \mathcal{P} \ : \ x = max(S, \succ_i)\}$. Note that λ can assign negative weights to preferences. Once we obtain this signed weight function λ, let $\succ$ be the collection of preferences that receive positive weights, and let $\triangleright'$ be the collection of preferences that receive negative weights. Let $\triangleright$ be the collection of the inverse of the preferences in $\triangleright'$. Finally, let λ^* be the weight function obtained from λ by assigning the absolute value of the weights assigned by λ. It directly follows that p is prudential with respect to the RPM $\langle \succ, \triangleright, \lambda^* \rangle$. We first introduce some notation and present crucial observations to construct the desired signed weight function λ.

Let p be a given RCF and Let $q : X \times \Omega \rightarrow \mathbb{R}$ be a mapping such that for each $S \in \Omega$ and $a \notin S$, $q(a, S) = q(a, S \cup \{a\})$ holds. Next, we present a result that is directly obtained by applying the *Möbius inversion*.[13]

Lemma 4 *For each choice set $S \in \Omega$, and alternative $a \in S$,*

$$p(a, S) = \sum_{S \subset T \subset X} q(a, T) \tag{4}$$

if and only if

$$q(a, S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} p(a, T) \tag{5}$$

Proof. For each alternative $a \in X$, note that $p(a, \cdot)$ and $q(a, \cdot)$ are real-valued functions defined on the domain consisting of all $S \in \Omega$ with $a \in S$. Then, by applying the Möbius inversion, we get the conclusion. ■

[13]See Stanley (1997), Section 3.7. See also Fiorini (2004), who makes the same observation.

Lemma 5 *For each choice set $S \in \Omega$ with $|S| = n-k$,*

$$\sum_{a \in X} |q(a,S)| \leq 2^k. \tag{6}$$

Proof. First, note that (6) can be written as follows:

$$\sum_{a \in S} |q(a,S)| + \sum_{b \notin S} |-q(b,S)| \leq 2^k. \tag{7}$$

For a set of real numbers,$\{x_1, x_2, \ldots x_n\}$, to show $\sum_{i=1}^{n} |x_i| \leq 2d$, it suffices to show that for each $I \subset \{1, 2, \cdots, n\}$, we have $-d \leq \sum_{i \in I} x_i \leq d$. Now, as the set of real numbers, consider $\{q(a,S)\}_{a \in X}$. It follows that to show that (7) holds, it suffices to show that for each $S_1 \subset S$ and $S_2 \subset X \setminus S$,

$$-2^{k-1} \leq \sum_{a \in S_1} q(a,S) - \sum_{b \in S_2} q(b,S) \leq 2^{k-1}$$

holds. To see this, first, for each $S_1 \subset S$ and $S_2 \subset X \setminus S$, it follows from Lemma 4 that for each $a \in S_1$ and for each $b \in S_2$, we have

$$q(a,S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} p(a,T) \text{ and } q(b,S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|-1} p(b,T). \tag{8}$$

Note that we obtain the second equality from Lemma 4, since for each $b \notin S$, by definition of $q(b,S)$, we have $q(b,S) = q(b, S \cup \{b\})$. Next, note that for each $T \in \Omega$ with $S \subset T$, $a \in S$, and $b \notin S$, $p(a,T)$ has the opposite sign of $p(b,T)$. Now, suppose for each $b \in S_2$, we multiply $q(b,S)$ with -1. Then, it follows from (8) that

$$\sum_{a \in S_1} q(a,S) - \sum_{b \in S_2} q(b,S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} \sum_{a \in S_1 \cup S_2} p(a,T). \tag{9}$$

Note that, for each $T \in \Omega$ such that $S \subset T$, $\sum_{a \in S_1 \cup S_2} p(a,T) \in [0,1]$. Therefore, the term $(-1)^{|T|-|S|} \sum_{a \in S_1 \cup S_2} p(a,T)$ adds at most 1 to the right-hand side of (9) if $|T| - |S|$ is even, and at least -1 if $|T| - |S|$ is odd. Since $|S| = n-k$, for each m with $n-k \leq m \leq n$, there are $\binom{k}{m-n+k}$ possible choice sets $T \in \Omega$ such that $S \subset T$ and $|T| = m$. Moreover, for each $i \in \{1, \ldots, k\}$, there are $\binom{k}{i}$ possible choice sets T such that $S \subset T$ and $|T| = n-k+i$. Now, the right-hand side of (9) reaches its maximum (minimum) when the negative (positive) terms are 0 and the positive (negative) terms are $1(-1)$. Thus, we get

$$-\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2i+1} \leq \sum_{S \subset T \subset X} (-1)^{|T|-|S|} \sum_{a \in S_1 \cup S_2} p(a,T) \leq \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i}.$$

It follows from the *binomial theorem* that both leftmost and rightmost sums are equal to 2^{k-1}. This, combined with (9), implies

$$-2^{k-1} \leq \sum_{a \in S_1} q(a, S) - \sum_{b \in S_2} q(b, S) \leq 2^{k-1}.$$

Then, as argued before, it follows that $\sum_{a \in X} |q(a, S)| \leq 2^k$. ■

Now, we are ready to complete the proof of Theorem 1. Recall that we assume $|X| = n$. For each $k \in \{1, \ldots, n\}$, let $\Omega_k = \{S \in \Omega \ : \ |S| > n - k\}$. Note that $\Omega_n = \Omega$ and $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n$. For each pair of preferences $\succ_1, \succ_2 \in \mathcal{P}$, $\succ_1$ is **k-identical** to $\succ_2$, denoted by $\succ_1 \sim_k \succ_2$, if the first k-ranked alternatives are the same. Note that $\sim_k$ is an equivalence relation on $\mathcal{P}$. Let $\mathcal{P}^k$ be the collection of preferences, such that each set (equivalence class) contains preferences that are k-identical to each other ($\mathcal{P}^k$ is the quotient space induced from $\sim_k$). For each $k \in \{1, \ldots, n\}$, let $[\succ^k]$ denote an **equivalence class** at $\mathcal{P}^k$, where $\succ^k$ linearly orders a fixed set of k alternatives in X.

Note that for each $k \in \{1, \ldots, n\}$, $S \in \Omega_k$ and $\succ_1, \succ_2 \in \mathcal{P}$, if $\succ_1 \sim_k \succ_2$, then since S contains more than $n - k$ alternatives, $\max(\succ_1, S) = \max(\succ_2, S)$. Therefore, for each $S \in \Omega_k$, it is sufficient to specify the weights on the equivalence classes contained in $\mathcal{P}^k$ instead of all the weights over $\mathcal{P}$. Let p_k be the restriction of p to Ω_k. Similarly, if λ is a signed weight function over $\mathcal{P}$, then let λ^k be the restriction of λ to $\mathcal{P}^k$, i.e. for each $[\succ^k] \in \mathcal{P}^k$, $\lambda^k[\succ^k] = \sum_{\succ_i \in [\succ^k]} \lambda(\succ_i)$. It directly follows that λ represents p if and only if for each $k \in \{1, \ldots, n\}$, λ^k represents p_k. In what follows, we inductively show that for each $k \in \{1, \ldots, n\}$, there is a signed weight function λ^k over $\mathcal{P}^k$ that represents p_k. For $k = n$ we obtain the desired λ.

For $k = 1$, $\Omega_1 = \{X\}$ and $\mathcal{P}^1$ consists of n-many equivalence classes such that each class contains all the preferences that top rank the same alternative, irrespective of whether these are chosen with positive probability. That is, if $X = \{x_1, \ldots, x_n\}$, then $\mathcal{P}^1 = \{[\succ^{x_1}], \cdots, [\succ^{x_n}]\}$, where for each $i \in \{1, \ldots, n\}$ and $\succ_i \in [\succ^{x_i}]$, $\max(X, \succ_i) = x_i$. Now, for each $x_i \in X$, define $\lambda^1([\succ^{x_i}]) = p(x_i, X)$. It directly follows that λ^1 is a signed weight function over $\mathcal{P}^1$ that represents p_1.

For $k = 2$, $\Omega_2 = \{X\} \cup \{X \setminus \{x\}\}_{x \in X}$ and $\mathcal{P}^2$ consists of $\binom{n}{2}$-many equivalence classes such that each class contains all the preferences that top rank the same two alternatives. Now, for each $[\succ_i^2] \in \mathcal{P}^2$ such that x_{i1} is the first-ranked alternative and x_{i2} is the second-ranked alternative, define $\lambda^2([\succ_i^2]) = p(x_{i2}, X \setminus \{x_{i1}\}) - p(x_{i2}, X)$. It directly follows that λ^2 is a signed weight function over $\mathcal{P}^2$ that represents p_2. Next, by our inductive hypothesis, we assume that for each $k \in \{1, \ldots, n-1\}$, there is a signed weight function λ^k over $\mathcal{P}^k$ that represents p_k. Next, we show that we can construct λ^{k+1} over $\mathcal{P}^{k+1}$ that represents p_{k+1}.

Note that $\mathcal{P}^{k+1}$ is a refinement of $\mathcal{P}^k$, in which each equivalence class $[\succ^k] \in \mathcal{P}^k$ is divided into sub-equivalence classes $\{[\succ_1^{k+1}], \cdots [\succ_{n-k}^{k+1}]\} \subset \mathcal{P}^{k+1}$. Given λ^k, we require λ^{k+1} satisfy for each $[\succ^k] \in \mathcal{P}^k$ the following

$$\lambda^k([\succ^k]) = \sum_{j=1}^{n-k} \lambda^{k+1}([\succ_j^{k+1}]). \tag{10}$$

If λ^{k+1} satisfies (10), then since induction hypothesis implies that λ^k represents p_k, we get for each $S \in \Omega_k$ and $x \in S$, $p(x,S) = \lambda^{k+1}\big(\{[\succ_j] \in \mathcal{P}^{k+1} \ : x = max(S, \succ_j)\}\big)$.

Next, we show that λ^{k+1} can be constructed such that (10) holds, and for each $S \in \Omega_{k+1} \backslash \Omega_k$, λ^{k+1} represents $p_{k+1}(S)$. To see this, pick any $S \in \Omega_{k+1} \backslash \Omega_k$. It follows that $|S| = n-k$. Let $S = \{x_1, , .., x_{n-k}\}$ and $X \setminus S = \{y_1, y_2, \cdots y_k\}$. Recall that each $[\succ^k] \in \mathcal{P}^k$ linearly orders a fixed set of k-many alternatives. Let $\{\succ^k\}$ denote the set of k alternatives ordered by $\succ^k$. Now, there exist $k!$-many $[\succ^k] \in \mathcal{P}^k$ such that $\{\succ^k\} = X \setminus S$. Let $\big\{[\succ_1^k], \cdots, [\succ_{k!}^k]\big\}$ be the collection of all such classes. Each preference that belongs to one of these classes is a different ordering of the same set of k alternatives.

Now, let $I = \{1, \ldots, k!\}$ and $J = \{1, \ldots, n-k\}$. For each $i \in I$ and $j \in J$, suppose that $\succ_{ij}^{k+1}$ linearly orders $X \setminus S$ as in $\succ_i^k$ and ranks x_j in the $k+1^{th}$ position. Consider the associated equivalence class $[\succ_{ij}^{k+1}]$. Next, we specify $\lambda^{k+1}([\succ_{ij}^{k+1}])$, the signed weight of $[\succ_{ij}^{k+1}]$, such that the resulting λ^{k+1} represents p_{k+1}. To see this, we proceed in two steps.

Step 1: First, we show that for each $S \in \Omega_{k+1} \backslash \Omega_k$, if the associated $\{\lambda_{ij}^{k+1}\}_{ij \in I \times J}$ satisfies the following two equalities for each $i \in I$ and $j \in J$,

$$\sum_{j \in J} \lambda_{ij}^{k+1} = \lambda^k([\succ_i^k]) \tag{RS}$$

$$\sum_{i \in I} \lambda_{ij}^{k+1} = q(x_j, S) \tag{CS}$$

then λ^{k+1} represents $p_{k+1}(S)$. For each $S \in \Omega$ and $x_j \in S$, $q(x_j, S)$ is as defined in (5) by using the given RCF p.

For each $S \in \Omega$ and $a \in S$, let $B(a, S)$ be the collection of all preferences at which a is the best alternative in S, and for each $k \in \mathbb{N}$ such that $n - k \leq |S|$, $\mathbf{B}^{k+1}(a, S)$ be the set of associated equivalence classes in $\mathcal{P}^{k+1}$, i.e. $B(a, S) = \{\succ \in \mathcal{P} : a = max(S, \succ)\}$ and $\mathbf{B}^{k+1}(a, S) = \{[\succ^{k+1}] \in \mathcal{P}^{k+1} : [\succ^{k+1}] \subset B(a, S)\}$. To prove the result we have to show that for each $x_j \in S$,

$$p(x_j, S) = \sum_{\{[\succ^{k+1}] \in \mathbf{B}^{k+1}(x_j, S)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{11}$$

To see this, for each $\succ \in \mathcal{P}$ and $a \in X$, let $W(\succ, a)$ denote the set of alternatives that are worse than a at $\succ$ and a itself, i.e. $W(\succ, a) = \{x \in X : a \succ x\} \cup \{a\}$. For each $S \in \Omega$ with $a \in X$. Let $Q(a, S)$ be the collection of all preferences such that $W(\succ, a)$ is exactly $S \cup \{a\}$ and for each $k \in \mathbb{N}$ such that $n - k \leq |S|$, $\mathbf{Q}^{k+1}(a, S)$ be the set of associated equivalence classes in $\mathcal{P}^{k+1}$, i.e. $Q(a, S) = \{\succ \in \mathcal{P} : W(\succ, a) = S \cup \{a\}\}$ and $\mathbf{Q}^{k+1}(a, S) = \{[\succ^{k+1}] \in \mathcal{P}^{k+1} : [\succ^{k+1}] \subset Q(a, S)\}$. Note that, for each $x_j \in S$, we have $Q(x_j, S) = \bigcup_{i \in I} [\succ_{ij}^{k+1}]$. Moreover, it directly follows from the definitions of $Q(x_j, \cdot)$ and $B(x_j, \cdot)$ that

$$B(x_j, S) = \bigcup_{S \subset T} Q(x_j, T). \tag{12}$$

It follows from this observation that the right-hand side of (11) can be written as

$$\sum_{S \subset T} \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, T)\}} \lambda^{k+1}([\succ^{t+1}]). \tag{13}$$

i. Since (CS) holds, we have

$$q(x_j, S) = \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, S)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{14}$$

ii. Next, we argue that for each $T \in \Omega$ such that $S \subsetneq T$,

$$q(x_j, T) = \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, T)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{15}$$

To see this, recall that by definition of $q(x_j, T)$ (5), we have

$$q(x_j, T) = \sum_{T \subset T'} (-1)^{|T'|-|T|} p(x_j, T'). \tag{16}$$

Since by the induction hypothesis, λ^k represents p_k, we have

$$p(x_j, T') = \sum_{\{[\succ^k] \in \mathbf{B}^k(x_j, T')\}} \lambda^k([\succ^k]). \tag{17}$$

Next, suppose that we substitute (17) into (16). Now, consider the set collection $\{B(x_j, T')\}_{\{T \subset T'\}}$. Note that if we apply the *principle of inclusion-exclusion* to this set collection, then we obtain $Q(x_j, T)$. It follows that

$$\sum_{T \subset T'} (-1)^{|T'|-|T|} \sum_{\{[\succ^k] \in \mathbf{B}^k(x_j, T')\}} \lambda^k([\succ^k]) = \sum_{\{[\succ^k] \in \mathbf{Q}^k(x_j, T)\}} \lambda^k([\succ^k]). \tag{18}$$

Since (RS) holds, we have

$$\sum_{\{[\succ^k] \in \mathbf{Q}^k(x_j, T)\}} \lambda^k([\succ^k]) = \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, T)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{19}$$

Thus, if we combine (16)-(19), then we obtain that (15) holds.

Now, (13) combined with (14) and (15) imply that the right-hand side of (11) equals to $\sum_{S \subset T} q(x_j, T)$. Finally, it follows from Lemma 4 that

$$p(x_j, S) = \sum_{S \subset T} q(x_j, T). \tag{20}$$

Thus, we obtain that (11) holds.

In what follows we show that for each $S \in \Omega_{k+1} \backslash \Omega_k$, there exists $k! \times (n-k)$ matrix $\lambda = [\lambda_{ij}^{k+1}]$ such that both (RS) and (CS) holds, and each $\lambda_{ij}^{k+1} \in [-1, 1]$. To prove this we use Lemma 3. For this, for each $i \in I$ let $r_i = \lambda^k([\succ_i^k])$ and for each $j \in J$ let $c_j = q(x_j, S)$. Then, let $R = [r_1, \ldots, r_{k!}]$ and $C = [c_1, \ldots, c_{n-k}]$. In Step 2, we show that the sum of C equals the sum of R. In Step 3, we show that for each $k > 1$, $2k! \geq \sum_{i=1}^{k!} |r_i| + \sum_{j=1}^{n-k} |c_j|$.

Step 2: We show that the sum of C equals the sum of R, i.e.

$$\sum_{j \in J} q(x_j, S) = \sum_{i \in I} \lambda^k [\succ_i^k]. \tag{21}$$

First, if we substitute (5) for each $q(x_j, S)$, then we get

$$\sum_{j\in J} q(x_j, S) = 1 + \sum_{j\in J} \sum_{S\subsetneq T} (-1)^{|T|-|S|} p(x_j, T). \tag{22}$$

Now, let $F(x_j)$ be the collection of preferences $\succ$ such that there exists $T \in \Omega$ such that $S \subsetneq T$ and x_j is the $\succ$-best alternative in T, i.e. $F(x_j) = \{\succ \in \mathcal{P} \ : \ max(T, \succ) = x_j \text{ for some } S \subsetneq T\}$. For each $k \in \mathbb{N}$ such that $n - k \leq |S|$, let $\mathbf{F}(x_j)$ be the set of associated equivalence classes in $\mathcal{P}^k$. Next, we show that for each $x_j \in S$,

$$\sum_{S\subsetneq T} (-1)^{|T|-|S|+1} p(x_j, T) = \sum_{\{[\succ^k]\in \mathbf{F}(x_j)\}} \lambda^k([\succ^k]). \tag{23}$$

To see this, first, since by the induction hypothesis, λ^k represents p_k, we can replace each $p(x_j, T)$ with $\sum_{\{[\succ^k]\in \mathbf{B}^k(x_j,T)\}} \lambda^k([\succ^k])$. Next, consider the set collection $\{B(x_j, T)\}_{\{S\subsetneq T\}}$. Since $\cup_{\{S\subsetneq T\}} B(x_j, T) = F(x_j)$, it follows from the *principle of inclusion-exclusion* that (23) holds. Next, when we substitute (23) in (22), we obtain

$$\sum_{j\in J} q(x_j, S) = 1 - \sum_{\{[\succ^k]\in \mathbf{F}(x_j)\}} \lambda^k([\succ^k]). \tag{24}$$

Then, since, by the induction hypothesis, λ^k represents p_k, we can replace 1 with $\sum_{\{[\succ^k]\in \mathcal{P}^k\}} \lambda^k([\succ^k])$. Finally, note that an equivalence class $[\succ^k] \notin \cup_{j\in J} \mathbf{F}(x_j)$ if and only if $\{\succ^k\} \cap S = \emptyset$. This means $\mathcal{P}^k \setminus \cup_{j\in J} \mathbf{F}(x_j) = \{[\succ_i^k]\}_{\{i\in I\}}$. It follows that (21) holds.

Step 3: To show that the base of induction holds, we showed that for $k = 1$ and $k = 2$, the desired signed weight functions exist. To get the desired signed weight functions for each $k + 1 > 2$, we will apply Lemma 3. To apply Lemma 3, we have to show that for each $k \geq 2$, $\sum_{i=1}^{k!} |r_i| + \sum_{j=1}^{n-k} |c_j| \leq 2k!$. In what follows we show that this is true. That is, we show that for each $S \in \Omega_{k+1} \setminus \Omega_k$

$$\sum_{i\in I} |\lambda^k([\succ_i^k])| + \sum_{j\in J} |q(x_j, S)| \leq 2k!. \tag{25}$$

To see this, first we will bound the term $\sum_{i\in I} |\lambda^k([\succ_i^k])|$. As noted before, each $i \in I = \{1, \ldots, k!\}$ corresponds to a specific linear ordering of $X \setminus S$. For each $y \notin S$, there are $k - 1!$ such different orderings that rank y at the k^{th} position. So, there are

$k-1!$ different equivalence classes in $\mathcal{P}^k$ that rank y at the k^{th} position. Let $I(y)$ be the index set of these equivalence classes. Since $\{I(y)\}_{y\notin S}$ partitions I, we have

$$\sum_{i\in I}|\lambda^k([\succ_i^k])| = \sum_{y\notin S}\sum_{i\in I(y)}|\lambda^k([\succ_i^k])|. \tag{26}$$

Now, fix $y \notin S$ and let $T = S\cup\{y\}$. Since for each $i \in I(y)$, $[\succ_i^k] \in \mathbf{Q}^k(y,T)$ and vice versa, we have

$$\sum_{i\in I(y)}|\lambda^k([\succ_i^k])| = \sum_{[\succ_i^k]\in\mathbf{Q}^k(y,T)}|\lambda^k([\succ_i^k])|. \tag{27}$$

Recall that by the definition of $q(y,T)$, we have

$$q(y,T) = \sum_{[\succ_i^k]\in\mathbf{Q}^k(y,T)}\lambda^k([\succ_i^k]). \tag{28}$$

Next, consider the construction of the values $\{\lambda^k([\succ_i^k]\}_{\{i\in I(y)\}}$ from the previous step. For $k=2$, as indicated in showing the base of induction, there is only one row; that is, there is a single $\{[\succ_i^k]\} = \mathbf{Q}^k(y,T)$. Therefore, we directly have $|\lambda^k([\succ_i^k])| = |q(y,T)|$. For $k>2$, we construct λ^k by applying Lemma 3. It follows from iii of Lemma 3 that

$$\sum_{[\succ_i^k]\in\mathbf{Q}^k(y,T)}|\lambda^k([\succ_i^k])| \le |q(y,T)| + \frac{(k-1)!}{n-k+1}. \tag{29}$$

Now, if we sum (29) over $y\notin S$, we get

$$\sum_{y\notin S}\sum_{[\succ_i^k]\in\mathbf{Q}^k(y,S\cup y)}|\lambda^k([\succ^k])| \le \left(\sum_{y\notin S}|q(y,S\cup y)|\right) + \frac{k!}{n-k+1}. \tag{30}$$

Recall that by definition, we have $\mathbf{Q}^k(y,S\cup y) = \mathbf{Q}^k(y,S)$ and $q(y,S\cup y) = q(y,S)$. Similarly, since each $j\in J=\{1,\dots,n\}$ denotes an alternative $x_j\in S$, we have $\sum_{x\in S}|q(x,S)| = \sum_{j\in J}|q(x_j,S)|$. Now, if we add $\sum_{j\in J}|q(x_j,S)|$ to both sides of (30), then we get

$$\sum_{i\in I}|\lambda^k([\succ_i^k])| + \sum_{j\in J}|q(x_j,S)| \le \sum_{x\in X}|q(x,S)| + \frac{k!}{n-k+1}. \tag{31}$$

Since by Lemma 5, $\sum_{x\in X}|q(x,S)| \le 2^k$, we get

$$\sum_{i\in I}|\lambda^k([\succ_i^k])| + \sum_{j\in S}|q(x_j,S)| \le 2^k + \frac{k!}{n-k+1}. \tag{32}$$

Finally, note that since for each k such that $2 < k < n$ $2^k \leq \frac{(2n-2k+1)k!}{n-k+1}$ holds, we have $2^k + \frac{k!}{n-k+1} \leq 2k!$. This, together with (32), implies that (25) holds. Thus, we complete the inductive construction of the desired signed weight function λ. This completes the proof. ■

B Proof of Proposition 1

Let $\langle \succ, \triangleright, \lambda \rangle$ and $\langle \succ', \triangleright', \lambda' \rangle$ be two RPMs that represent the same RCF p. Now, for each choice set $S \in \Omega$, both λ and λ' should satisfy the identity (CS) used in Step 1 of the proof of Theorem 1. That is, for each $S \in \Omega$ and $x \in S$ both λ and λ' generates the same $q(x, S)$ value. Therefore, if we can show that $\lambda(x = B_k|S, \succ)$ can be expressed in terms of $q(x, \cdot)$, then (1) follows. To see this, let $\langle \succ, \triangleright, \lambda \rangle$ be any RPM that represents p. Next, for each $S \in \Omega$, $x \in S$, and $k \in \{1, \ldots, |S|\}$, consider a partition (S_1, S_2) of S such that $x \in S_2$ and $|S_1| = k-1$. Let $\mathbb{P}(S, x, k)$ be the collection of all these partitions. Now, for each fixed $(S_1, S_2) \in \mathbb{P}(S, x, k)$, let $\lambda(x|S_1, S_2, \succ)$ be the sum of the weights of the pro-preferences at which x is the best alternative in S_2 and the worst alternative in S_1. Note that for each such pro-preference, x is the k^{th}-best alternative in S. Similarly, let $\lambda(x|S_1, S_2, \triangleright)$ be the sum of the weights of the con-preferences at which x is the best alternative in S_1 and the worst alternative in S_2. Note that for each such con-preference x is the k^{th}-worst alternative in S. Now, it follows that we have:

$$\lambda(x = B_k|S, \succ) = \sum_{\{(S_1,S_2) \in \mathbb{P}(S,x,k)\}} \lambda(x|S_1, S_2, \succ), \tag{33}$$

$$\lambda(x = W_k|S, \triangleright) = \sum_{\{(S_1,S_2) \in \mathbb{P}(S,x,k)\}} \lambda(x|S_1, S_2, \triangleright). \tag{34}$$

Since for each $T \in \Omega$ such that $S_2 \subset T$ and $T \subset X \setminus S_1$, by definition, $q(x, T)$ gives the difference between the sum of the weights of the pro-preferences at which x is the best alternative in S and sum of the weights of the con-preferences at which x is the worst alternative in S, it follows that

$$\sum_{\mathbb{P}(S,x,k)} \lambda(x|S_1, S_2, \succ) - \sum_{\mathbb{P}(S,x,k)} \lambda(x|S_1, S_2, \triangleright) = \sum_{\mathbb{P}(S,x,k)} \sum_{S_2 \subset T \subset X \setminus S_1} q(x, T). \tag{35}$$

Finally, if we substitute (33) and (34) in (35), then we express $\lambda(x = B_k|S, \succ) - \lambda(x = B_k|S, \triangleright)$ only in terms of $q(x, \cdot)$, as desired. ■

C Sufficient conditions for total unimodularity

Lemma 6 (Heller & Tompkins (1956)) *Let A be an $m \times n$ matrix whose rows can be partitioned into two disjoint sets R_1 and R_2. Then, A is totally unimodular if:*

1. *Each entry in A is 0, 1, or -1;*
2. *Each column of A contains at most two non-zero entries;*
3. *If two non-zero entries in a column of A have the same sign, then the row of one is in R_1, and the other is in R_2;*
4. *If two non-zero entries in a column of A have opposite signs, then the rows of both are in R_1, or both in R_2.*

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