Choice through a unified lens: The prudential model*

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Abstract

We present a new choice model. An agent is endowed with two sets of preferences: pro-preferences and con-preferences. For each choice set, if an alternative is the best (worst) for a *pro-preference* (*con-preference*), then this is a *pro* (*con*) for choosing that alternative. The alternative with more pros than cons is chosen from each choice set. Each preference may have a *weight* reflecting its salience. In this case, the probability that an alternative is chosen equals the difference between the weights of its pros and cons. We show that this model provides a unified lens through which every nuance of the rich human choice behavior can be structurally explained.

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Contents

1 Introduction

Charles Darwin, the legendary naturalist, wrote "The day of days!" in his journal on November 11, 1838, when his cousin Emma Wedgwood accepted his marriage proposal. However, whether to marry at all had been a hard decision for Darwin. Just a few months prior, Darwin had scribbled a carefully considered list of *pros* –such as "constant companion", "charms of music", "female chit-chat"–and *cons* –such as "may be quarrelling", "fewer conversations with clever people", "no books"– regarding the potential impact of marriage on his life.[1] With this list of pros and cons, Darwin seems to follow a choice procedure ascribed to Benjamin Franklin.[2] Here we present Franklin (1887)'s choice procedure in his own words.

> *To get over this, my Way is, to divide half a Sheet of Paper by a Line into two Columns, writing over the one Pro, and over the other Con. I endeavour to estimate their respective Weights; and where I find two, one on each side, that seem equal, I strike them both out: If I find a Reason pro equal to some two Reasons con, I strike out the three. If I judge some two Reasons con equal to some three Reasons pro, I strike out the five; and thus proceeding I find at length where the Ballance lies. And tho' the Weight of Reasons cannot be taken with the Precision of Algebraic Quantities, yet when each is thus considered separately and comparatively, and the whole lies before me, I think I can judge better, and am less likely to take a rash Step; and in fact I have found great Advantage from this kind of Equation, in what may be called Moral or Prudential Algebra.*

Choice models most commonly used in economics are based on maximization of preferences. An alternative mode of choice, which is common for the scholarly work in other social disciplines such as history, law, and political science, is the less

[1] See Glass (1988) for the full list.

[2] In 1772, a man named Joseph Priestley wrote a letter to Benjamin Franklin asking for Franklin's advice on a decision he was trying to make. Franklin wrote back indicating that he could not tell him what to do, but he could tell him how to make his decision, and suggested his *prudential algebra*.

formal *reason-based analysis* (Shafir et al. (1993)). Reason-based analysis is also commonly used for the analysis of 'case studies' in business and law schools. In the vein of Franklin's prudential algebra, first, various arguments that support or oppose an alternative are identified, then the balance of these arguments determines the choice.[3] We formulate and analyze the *prudential choice model* that connects these two approaches by presenting a reason-based choice model, in which the 'reasons' are formed by using a preference-based language.

In the last decade several choice models have been proposed to accommodate choice behavior that classical theories fail to explain. In this study we observe that every nuance of the rich human choice behavior can be captured via a structured model that exhibits limited context-dependency. In our analysis and examples we aim to illustrate that prudential model provides a canonical language to address several questions in choice theory. The random prudential model renders an *additive* and *structurally-invariant* representation, similar to the random utility model. This reflects itself as a form of uniqueness in the representation, which facilitates the model's identification. In our examples, we present specific prudential models that accommodate observed choice behavior that commonly used models fail to explain–such as *similarity effect* and *attraction effect*–by capturing the key aspects of the contexts in which these choice patterns are observed.

First, we formulate the prudential choice model in the deterministic choice setup by extending Franklin's prudential algebra to choice sets that possibly contain more than two alternatives. A *choice function* C singles out an alternative from each *choice set* S, which is a nonempty subset of the grand *alternative set* X. A *(deterministic) prudential model* (PM) is a pair $\langle \succ, \triangleright \rangle$ such that $\succ = \{\succ_1, \cdots, \succ_m\}$ is a collection of *pro-preferences*[4] and $\triangleright = \{\triangleright_1, \cdots, \triangleright_q\}$ is a collection of *con-preferences*. Given an PM $\langle \succ, \triangleright \rangle$, for each choice set S and alternative x, if x is the $\succ_i$-best alternative in

[3] Shafir et al. (1993) argue that reason-based analyses have been used to understand unique historic, legal and political decisions. Examples include presidential decisions taken during the Cuban missile crisis (Allison (1971)), the Camp David accords (Telhami (1990)), and the Vietnam war (Gelb & Betts (2016)).

[4] A preference is a complete, transitive, and antisymmetric binary relation on X.

S for some $\succ_i \in \succ$, then we interpret this as a 'pro' for choosing x from S. On the other hand, if x is the $\triangleright_i$-worst alternative in S for some $\triangleright_i \in \triangleright$, then we interpret this as a 'con' for choosing x from S. More formally, $Pros(x, S)$ denotes the set of pro-preferences ($\succ_i \in \succ$) at which x is the best alternative in S and $Cons(x, S)$ denotes the set of con-preferences ($\triangleright_i \in \triangleright$) at which x is the worst alternative in S. Our central new concept is the following: A choice function is *prudential* if there is an PM $\langle \succ, \triangleright \rangle$ such that for each choice set S, an alternative x is chosen from S if and only if the number of $Pros(x, S)$ is greater than the number of $Cons(x, S)$.

Next, we formulate the prudential model in the stochastic choice setup. In this setup, an agent's repeated choices or a group's choices are summarized by a *random choice function* (RCF) p, which assigns to each choice set S, a probability measure over S. For each choice set S and alternative x, we denote by $p(x, S)$ the probability that alternative x is chosen from choice set S. A *random prudential model* (RPM) is a triplet $\langle \succ, \triangleright, \lambda \rangle$, where $\succ$ and $\triangleright$ stand for pro-preferences and con-preferences, as before. The weight function λ assigns to each pro-preference $\succ_i \in \succ$ and con-preference $\triangleright_i \in \triangleright$, a value in the $(0, 1]$ interval, which we interpret as a measure of the salience of each preference. In line with the experimental findings of Shafir (1993) indicating that the weight assigned to the pros is more than the weight assigned to the cons, we require that the difference between the weighted sum of pro-preferences and con-preferences is unity. An RCF p is *prudential* if there is an RPM $\langle \succ, \triangleright, \lambda \rangle$ such that for each choice set S and alternative x,

$$p(x, S) = \lambda(Pros(x, S)) - \lambda(Cons(x, S)),$$

where $\lambda(Pros(x, S))$ and $\lambda(Cons(x, S))$ are the sum of the weights over $Pros(x, S)$ and $Cons(x, S)$.

The most familiar stochastic choice model in economics is the *random utility model* (RUM), which assumes that an agent is endowed with a probability measure μ over a set of preferences $\succ$ such that he randomly selects a preference to be maximized from $\succ$ according to μ. Each RUM $\langle \succ, \mu \rangle$ is an RPM in which there is no set of con-preferences. As for the similarity between the RPM and the RUM, both models

are *additive*, in the sense that the choice probability of an alternative is calculated by summing up the weights assigned to the preferences. The primitives of both the RPM and RUM are structurally invariant, in the sense that the decision maker uses the same $\langle \succ, \mu \rangle$ and $\langle \succ, \triangleright, \lambda \rangle$ to make a choice from each choice set. These two features of RUM brings stringency in its identification, which reflects itself in its characterization.[5] On the other hand, despite the similarity between the RPM and the RUM, in our Theorem 1, we show that every random choice function is prudential. Then, by using the construction in Theorem 1's proof–based on an original extension of Ford Jr & Fulkerson (2015)'s seminal result in optimization theory–and two key results from the integer-programming literature, we show that each (deterministic) choice function is prudential.[6]

Our main results show that the prudential model–an additive model similar to the RUM–provides a canonical language to describe any choice behavior in terms of structurally-invariant primitives. We believe that being inclusive does not take away from the relevance of a structured model, but opens up new directions to pursue.[7] It seems that what makes a choice model economically interesting is twofold. One concern is whether the primitives of the model can be precisely identified from the observed choices. The other concern is whether the model provides plausible explanations for observed choice patterns that classical models fail to explain. In the rest of the paper, we aim to address these concerns. Besides, our results facilitate identifi-

[5]Namely, the RCFs that render a random utility representation are those with nonnegative Block-Marschak polynomials. See Block & Marschak (1960), Falmagne (1978), McFadden (1978), and Barberá & Pattanaik (1986).

[6]This result does not directly follow from Theorem 1, since a prudential model is not a direct adaptation of the random prudential model, in that we require each preference to have a fixed unit weight instead of having fractional weights. To best of our knowledge the use of integer programming techniques in this context is new.

[7]As a thought experiment, consider the most commonly used random choice model in economics, namely the Luce rule, and imagine that the Luce rule is permissive enough to accommodate every choice behavior. This would not make the Luce rule useless, but may make it even more appealing, since no data needs to be eliminated in empirical applications, and there would not be any model misspecification problem.

cation of other inclusive choice models, which otherwise may be rather difficult. We present an application along these lines, in which we show that each choice function is *plurality-rationalizable*.[8]

As for the identification of the primitives from observed choices, the RPM has characteristics similar to the RUM. In general, an RCF may have different random utility representations even with disjoint sets of preferences. However, Falmagne (1978) argues that random utility representation is essentially unique, in the sense that the sum of the probabilities assigned to the preferences at which an alternative x is the kth-best in a choice set S is the same for each random utility representation of the given RCF. In the vein of Falmagne's result, we show that for each RCF the difference between the sum of the weights assigned to the pro-preferences at which x is the kth-best alternative in S and the sum of the weights assigned to the con-preferences at which x is the kth-worst alternative in S is the same for each prudential representation of the given RCF.[9] Falmagne proposes an empirical identification method for RUMs based on his uniqueness result. We believe that our results and constructions in their proofs paves the way for a similar identification method for the RPM.

In Section 2.2, we present specific prudential choice models that accommodate observed choice behavior, such as the *similarity effect* and the *attraction effect* that commonly used choice models fail to explain. This may seem of little importance for an inclusive choice model, however, our point is to illustrate that tailored prudential choice models capture the key aspects of the contexts in which these choice patterns are observed. For example, in the classical attraction effect scenario it seems that there are only two relevant criteria for choice, such as price and quantity. The pro- and con-preferences used in our Example 3 correspond to these criteria. As a result, the choice probability of an alternative may increase when a *decoy* is added, since this alternative may no longer be the worst one according to a relevant criterion. A

[8] It turns out that a PM can also be viewed as a collective decision making model based on plurality voting. We present the model in Section 3.4. The model and the result can thought as a generalization of an earlier model and a related result by McGarvey (1953).

[9] In particular, both the RUM and the RPM render a unique representation when there are only three alternatives.

key feature that derives the similarity effect is that there are two distinct attributes that are relevant for choice, one of which is of major importance, whereas the other is of secondary importance.[10] The pro- and con-preferences used in our Example 2 reflects this logic. These examples indicate that analyzing prudential choice model with restricted pro- and con-preferences may lead to insightful results. In this vein, in Section 2.6, we consider choice problems in which there are two observable orderings of the alternatives that are relevant for choice. Then, we provide a set of choice axioms that guarantee the observed choices are generated via an RPM in which the pro-preferences and the con-preferences are obtained from the observed orderings.

1.1 Related literature

In the deterministic choice literature, previous choice models proposed by Kalai et al. (2002) and Bossert & Sprumont (2013) yield similar "anything goes" results. A choice function is *rationalizable by multiple rationales* (Kalai et al. (2002)) if there is a collection of preference relations such that for each choice set the choice is made by maximizing one of these preferences. Put differently, the decision maker selects a preference to be maximized for each choice set. A choice function is *backwards-induction rationalizable* (Bossert & Sprumont (2013)) if there is an extensive-form game such that for each choice set the backwards-induction outcome of the restriction of the game to the choice set coincides with the choice. In this model, for each choice set, a new game is obtained by pruning the original tree of all branches leading to unavailable alternatives. In the stochastic choice setup, Manzini & Mariotti (2014) provide an anything-goes result for the *menu-dependent random consideration set rules*. In this model, an agent keeps a single preference relation and attaches to each alternative a choice-set-specific attention parameter. Then, from each choice he chooses an alternative with the probability that no more-preferable alternative grabs his attention. In contrast to these models, we believe that the prudential model is more structured, and exhibits limited context dependency. In that, an agent follow-

[10] In Debreu's example Debreu (1960), whether it is a travel by bus or by train is the primary attribute, whereas the color of the bus is the secondary attribute.

ing a prudential model only restricts the pro-preferences and con-preferences to the given choice set to make a choice.

In the discrete-choice literature, there is a related line of research about the probabilistic models of *best-worst choices* (Marley & Louviere (2005)). It is assumed that an agent not only reports his best choice but also his worst one from each choice set. In contrast to the RPM, the existing models analyzed in this enriched framework lie within the random utility framework. In the social choice theory literature, Felsenthal (1989) proposes the approval-disapproval voting model as an extension of the commonly used approval voting model of Brams & Fishburn (1978). Similar to our prudential model, for a given preference profile each alternative gets a score that equals the difference between the number of voters that top rank the alternative and the number of voters that bottom rank the alternative. Then, the alternative(s) with the highest score is chosen. In contrast, we identify pro- and con preferences from agents' choices. Therefore, we could not see any direct implication of the existing results in voting theory to our analysis.

Our Theorem 1 is partly related to a result in a contemporary paper by Saito (2017), who offers characterizations of the mixed logit model. It follows from Proposition 3 of this paper, which is proved by using a different approach, that each RCF can be expressed as an affine combination of two random utility functions. This result renders a prudential representation without requiring weights be at most unity.[11] Besides the contemporaneous nature and different focus of the two papers, requiring weights be at most unity is critical for our results, and poses additional technical challenges. We present the detailed discussions in Remark 1 and Remark 2.

[11] Saito (2017) reports this observation in footnote 7 on p. 15. We are grateful to an anonymous referee for bringing this connection to our awareness.

2 Prudential random choice functions

2.1 The model

Given a nonempty finite alternative set X, any nonempty subset S is called a **choice set**. Let Ω denote the collection of all choice sets. A **random choice function** (RCF) p is a mapping that assigns each choice set $S \in \Omega$, a probability measure over S. For each $S \in \Omega$ and $x \in S$, we denote by $p(x, S)$ the probability that alternative x is chosen from choice set S. A *preference*, denoted generically by $\succ_i$ or $\triangleright_i$, is a complete, transitive, and antisymmetric binary relation on X.

A **random prudential model** (RPM) is a triplet $\langle \succ, \triangleright, \lambda \rangle$, where $\succ = \{\succ_1, \cdots, \succ_m\}$ and $\triangleright = \{\triangleright_1, \cdots, \triangleright_q\}$ are sets of pro- and con-preferences on X. We assume that if $\succ_i$ is a pro-preference, then there is no con-preference $\triangleright_i$ which is the inverse of $\succ_i$. That is, being best according to a preference should not simultaneously be a pro and con for an alternative. Finally, the **weight function**, denoted by λ is such that for each $\succ_i \in \succ$ and $\triangleright_i \in \triangleright$, we have $\lambda(\succ_i) \in (0, 1]$, $\lambda(\triangleright_i) \in (0, 1]$, and the difference between the weighted sum of pro-preferences and con-preferences is one, i.e. $\sum_{\{\succ_i \in \succ\}} \lambda(\succ_i) - \sum_{\{\triangleright_i \in \triangleright\}} \lambda(\triangleright_i) = 1$. The weight function λ acts like a probability measure over the set of preferences that can assign negative values. In measure theoretic language, the primitive of a random prudential model is a *signed probability measure* defined over the set of preferences.

Given an RPM $\langle \succ, \triangleright, \lambda \rangle$, for each choice set S and alternative $x \in S$, if x is the $\succ_i$-best alternative in S for some $\succ_i \in \succ$, then we interpret this as a 'pro' for choosing x from S. On the other hand, if x is the $\triangleright_i$-worst alternative in S for some $\triangleright_i \in \triangleright$, then we interpret this as a 'con' for choosing x from S. We interpret the weight assigned to each pro-preference or con-preference as a measure of the salience of that preference. To define when an RCF is prudential, let $Pros(x, S) = \{\succ_i \in \succ : \ x = max(S, \succ_i)\}$ and $Cons(x, S) = \{\triangleright_i \in \triangleright \ : \ x = min(S, \triangleright_i)\}$. Next, we formally define when a RCF is prudential.

Definition 1 *An RCF p is* ***prudential*** *if there is an RPM $\langle \succ, \triangleright, \lambda \rangle$ such that for each choice set $S \in \Omega$ and $x \in S$,*

$$p(x, S) = \lambda(Pros(x, S)) - \lambda(Cons(x, S)), \tag{1}$$

where $\lambda(Pros(x, S))$ and $\lambda(Cons(x, S))$ are the sum of the weights over $Pros(x, S)$ and $Cons(x, S)$.

As the reader would easily notice not every RPM $\langle \succ, \triangleright, \lambda \rangle$ yields an RCF. For this to be true, for each choice set $S \in \Omega$ and $x \in S$, expression in (1) should be nonnegative and sum up to one. These additional requirements are imposed on the model by our Definition 1. Next, we provide a less structured formulation of the prudential model that always yields an RCF. For a given RPM $\langle \succ, \triangleright, \lambda \rangle$, let for each $S \in \Omega$ and $x \in S$, $\lambda(x, S) = \lambda(Pros(x, S)) - \lambda(Cons(x, S))$, and $S^+ = \{x \in S : \lambda(x, S) > 0\}$. It directly follows that if an RCF p can be represented as in 1, then p can be represented as in 2. Since Definition 1 is a rather parsimonious one, converse does not follow directly. This parsimony derives the uniqueness result in Proposition 1, and paves the way for obtaining our Theorem 2.

Definition 2 *An RCF p is* ***prudential*** *if there is an RPM $\langle \succ, \triangleright, \lambda \rangle$ such that for each choice set $S \in \Omega$ and $x \in S$,*

$$p(x, S) = \begin{cases} \dfrac{\lambda(x, S)}{\sum_{\{y \in S^+\}} \lambda(y, S)} & \text{if } \lambda(x, S) > 0 \\ 0 & \text{if } \lambda(x, S) \leq 0 \end{cases} \tag{2}$$

That is, to make a choice from each choice set S, a prudential agent considers the alternatives with a positive $\lambda(x, S)$ score, and chooses each alternative from this consideration set with a probability proportional to its weight.

We can render an intuitive interpretation for the RPM in the vein of Tversky (1972)'s *elimination by aspects*, in which an agent views each alternative as a set of attributes and makes his choice by following a probabilistic process that eliminates alternatives based on their attributes.[12] To see the connection, consider a

[12] Tversky (1972) argues that elimination by aspects reflects the choice process followed by agents more precisely than the classical choice models.

con-preference $\triangleright_i$; if an alternative x is not the $\triangleright_i$-worst alternative in a choice set S, then say that x is *acceptable* according to $\triangleright_i$ in S. Now, we can interpret the statement "*x has attribute i in choice set S*" as "*x is acceptable according to $\triangleright_i$ in S*". Thus, for a given RPM, each alternative without attribute i in choice set S is eliminated with a probability proportional to the weight of attribute i. In line with this interpretation, we illustrate in our Example 2 and Example 3 that each preference in an RPM can be interpreted as an attribute or a relevant criterion for the choice. The agent's attitude to these criteria is different in that if it is a pro-preference, then he seeks maximization; if it is a con-preference, then he is satisfied by the elimination of the worst alternative.

Remark 1 We require the weights be at most unity for each pro- or con-preference. As for Definition 1, this is not a simple normalization exercise. To see this, suppose an RCF p is represented as in (1) where some weights are above one. If we divide all the terms with a common element, then (1) no longer holds. Besides technical challenges posed by requiring each weight be at most unity, it is critical for obtaining Proposition 1, Theorem 2, and understanding the implication of Block-Marschak polynomials being locally nonnegative.[13]

2.2 Examples

Following examples present specific prudential choice models that accommodate observed choice behavior that commonly used choice models fail to explain. As discussed in the introduction, this may seem of little importance for an inclusive choice model. However, our point is to illustrate that the tailored prudential choice models capture the key aspects of the contexts in which these choice patterns are observed. First, we present an example in which all preferences have a weight of one. Therefore, the resulting choice is deterministic and illustrates the deterministic prudential model.

[13]We explore this connection in Section 2.5.

Example 1 (Binary choice cycles) Suppose $X = \{x, y, z\}$ and consider the following RPM $\langle \succ, \triangleright, \lambda \rangle$. Note that x is chosen from the grand set and when compared to

(1)	(1)	(1)
$\succ_1$	$\succ_2$	$\triangleright_1$
x	y	z
y	z	x
z	x	y

y, y is chosen when compared to z, but z is chosen when compared to x. That is, the given PM generates the choice behavior of an agent who exhibits a binary choice cycle between x, y, z, and chooses x from the grand set.

Example 2 (Similarity Effect) Suppose $X = \{x_1, x_2, y\}$, where x_1 and x_2 are similar alternatives, such as recordings of the same Beethoven symphony by different conductors, while y is a distinct alternative, such as a Debussy suite. Suppose between any pair of the three recordings our classical music aficionado chooses with equal probabilities, and he chooses from the set $\{x_1, x_2, y\}$ with probabilities $0.25, 0.25$, and 0.5 respectively.[14] Consider the RPM $\langle \succ, \triangleright, \lambda \rangle$ presented below:

(1/4)	(1/4)	(1/2)	(1/2)
$\succ_1$ / $\triangleright_1$	$\succ_2$ / $\triangleright_2$	$\succ_3$	$\succ_4$
y	y	x_1	x_2
x_1	x_2	x_2	x_1
x_2	x_1	y	y

We choose $(\succ_1, \triangleright_1)$ and $(\succ_2, \triangleright_2)$ as the same preferences, and assign the same weight. In the story, the composer has primary importance, whereas the conductor has secondary importance. In line with this observation, all the preferences in the given

[14] Debreu (1960) proposes this example to highlight a shortcoming of the Luce rule (Luce (1959)). This phenomena is later referred to as the *similarity effect* or *duplicates effect*. See Gul et al. (2014) for a random choice model that accommodates the similarity effect.

RPM ranks the recordings first according to composer, then according to conductor. One can easily verify that the induced RCF generates our classical music aficionado's choices.

In Example 2, there are two alternatives that are slightly different. If the substitution is not extreme, then an agent may exhibit a choice pattern incompatible with the RUM. In this vein, the next example illustrates that when we introduce an asymmetrically dominated alternative, the choice probability of the dominating alternative may go up. This choice behavior, known as the *attraction effect*, is incompatible with any RUM.[15]

Example 3 (Attraction Effect) Suppose $X = \{x, y, z\}$, where x and y are two competing alternatives such that none clearly dominates the other, and z is another alternative that is dominated by x but not y. To illustrate the attraction effect, we follow the formulation in our Definition 2. Consider the following RPM $\langle \succ, \triangleright, \lambda \rangle$, in which there is single pair of preferences used both as the pro- and con-preferences. We can interpret this preference pair as two distinct criteria that order the alternatives.

$(1/2)$	$(1/2)$	$(1/4)$	$(1/4)$
$\succ_1$	$\succ_2$	$\triangleright_1$	$\triangleright_2$
x	y	x	y
z	x	z	x
y	z	y	z

Now, since for both criteria x is better than z, we get $p(x, \{x, z\}) = 1$. Since x and y fail to dominate each other, and y fail to dominate z, we get $p(y, \{x, y\}) = p(y, \{y, z\}) = 1/2$. That is, z is a 'decoy' for x when y is available. Note that when

[15]Experimental evidence for the attraction effect is first presented by Payne & Puto (1982) and Huber & Puto (1983). Following their work, evidence for the attraction effect has been observed in a wide variety of settings. For a list of these results, consult Rieskamp et al. (2006). On the theory side, Echenique et al. (2013) propose a Luce-type model and Natenzon (n.d.) proposes a learning model that accommodate the attraction effect in the random choice setup.

only x and y are available, since x is the $\triangleright_2$-worst alternative, x is eliminated with a weight of $1/2$,. However, when the decoy z is added to the choice set, then x is no longer the $\triangleright_2$-worst alternative, and we get $p(x, \{x, y, z\}) = 2/3$. That is, availability of decoy z increases the choice probability of x. Thus, the proposed RPM presents an attraction effect scenario. One can imagine several similar choice scenarios, in which the criteria that are relevant for choice –such as price and quality–are observable. In Section 2.6, we analyze the prudential random choice model specified for a given pair of preferences, which generalizes this example.

2.3 Main result

In our main result, we show that every random choice function is prudential. We present a detailed discussion of the result in the introduction. We present the proof in Section 5. As a notable technical contribution, we extend and use Ford-Fulkerson Theorem (Ford Jr & Fulkerson (2015)) from combinatorial matrix theory.[16] Next, we state the theorem and present an overview of the proof. Then, we discuss the technical connection to Saito (2017).

Theorem 1 *Every random choice function is prudential.*

Here, we present an overview of the proof. For a given RCF p, we show that there is a *signed weight function* λ, which assigns each preference $\succ_i$, a value $\lambda(\succ_i) \in [-1, 1]$ such that λ *represents* p. That is, for each choice set S and $x \in S$, $p(x, S)$ is the sum of the weights over preferences at which x is the top-ranked alternative. Once we obtain this signed weight function λ, let $\succ$ be the collection of preferences that receive positive weights, and $\triangleright$ be the collection of the inverses of the preferences that receive negative weights. Let λ^* be the weight function obtained from λ by assigning the absolute value of the weights assigned by λ. It directly follows that p is prudential with respect to the RPM $\langle \succ, \triangleright, \lambda^* \rangle$. Therefore, to prove the theorem, it is sufficient to show that there exists a signed weight function that represents p. We prove this by

[16] It is also known as *max-flow min-cut theorem* in optimization theory.

induction.

To clarify the induction argument, for $k = 1$, let $\Omega_1 = \{X\}$ and let $\mathcal{P}^1$ consists of n-many equivalence classes such that each class contains all the preferences that top rank the same alternative, irrespective of whether these are chosen with positive probability. That is, for $X = \{x_1, \ldots, x_n\}$, we have $\mathcal{P}^1 = \{[\succ^{x_1}], \cdots, [\succ^{x_n}]\}$, where for each $i \in \{1, \ldots, n\}$ and preference $\succ_i \in [\succ^{x_i}]$, $\max(X, \succ_i) = x_i$. Now for each $x_i \in X$, define $\lambda^1([\succ^{x_i}]) = p(x_i, X)$. It directly follows that λ^1 is a signed weight function over $\mathcal{P}^1$ that represents the restriction of the given RCF to Ω_1, denoted by p_1. By proceeding inductively, it remains to show that we can construct λ^{k+1} over $\mathcal{P}^{k+1}$ that represents p_{k+1}. In Step 1 of the proof we show that finding such a λ^{k+1} boils down to finding a solution to the system of equalities described by *row sums* (RS) and *column sums* (CS).[17] To get an intuition for (RS), while moving from the kth-step to the $(k+1)^{th}$-step, each $[\succ^k]$ is decomposed into a collection $\{[\succ_j^{k+1}]\}_{j \in J}$ such that for each $[\succ_j^{k+1}]$ there exists an alternative x_j that is not linearly ordered by $[\succ^k]$, but placed at $[\succ_j^{k+1}]$ right on top of the alternatives that are not linearly ordered by $[\succ^k]$. Therefore, the sum of the weights assigned to $\{[\succ_j^{k+1}]\}_{j \in J}$ should be equal to the weight assigned to $[\succ^k]$. This gives us the set of equalities formulated in (RS). To get an intuition for (CS), let S be the set of alternatives that are not linearly ordered by $[\succ^k]$. Now, we should design λ^{k+1} such that for each $x_j \in S$, $p(x_j, S)$ should be equal to the sum of the weights assigned to preferences at which x_j is the top-ranked alternative in S. The set of equalities formulated in (CS) guarantees this.[18]

Next, we observe that finding a solution to the system described by (RS) and (CS) can be translated to the following basic problem: Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be two real-valued vectors such that the sum of R equals to the sum of C. Now, for which R and C can we find an $m \times n$ matrix $A = [a_{ij}]$ such that A has row sum vector R and column sum vector C, and each entry $a_{ij} \in [-1, 1]$? Ford Jr & Fulkerson (2015) provide a full answer to this question when R and C are positive

[17] Up to this point the proof structure is similar to the one followed by Falmagne (1978) and Barberá & Pattanaik (1986) for the charaterization of RUM.

[18] A related key observation is our Lemma 6, which we obtain by using the *Mobius inversion*.

real valued.[19] However, a peculiarity of our problem is that the corresponding row and column values can be negative. Indeed, we get nonnegative-valued rows and columns only if the Block-Marschak polynomials are nonnegative, that is, the given p is a RUM. In our Lemma 5, we provide an extension of Ford Jr & Fulkerson (2015)'s result that paves the way for our proof.[20] Then, in Step 2 we show that (RS) equals (CS). In Step 3, by using a structural result presented in Lemma 7, we show that the row and column vectors associated with (RS) and (CS) satisfy the premises of our Lemma 5. This completes the construction of the desired signed weight function.

Remark 2 As discussed in Section 1.1, it follows from the results of a contemporary paper by Saito (2017) that each RCF can be expressed as an affine combination of two random utility functions. It follows from our Theorem 1 that the weights used in this affine combination can be chosen from $[-1, 1]$ interval. To see the technical difference, note that by following the construction in our proof and directly applying the the Ford-Fulkerson Theorem, without using several results that we obtain, each RCF can be expressed as an affine combination of random utility functions. On the other hand, to show that these weights can be chosen from $[-1, 1]$ interval, we extend the Ford-Fulkerson Theorem (see Lemma 5) and follow a deliberate induction argument supported by other structural results, such as Lemma 7.

2.4 Uniqueness

The primitives of the RUM model are structurally invariant in the sense that the agent uses the same $\succ$ and μ to make a choice from each choice set. This feature of the RUM brings precision in identifying the choice behavior. To elaborate on this, although an RCF may have different random utility representations even with disjoint sets of preferences, Falmagne (1978) argues that random utility representation is essentially unique. That is, the sum of the probabilities assigned to the preferences

[19] Brualdi & Ryser (1991) provides a detailed account of similar results.

[20] Roughly speaking, for extending the result to real-valued vectors, the sum of the absolute values of the rows and columns should respect a specific bound.

at which an alternative x is the kth-best in a choice set S is the same for all random utility representations of the given RCF. Similarly, the primitives of an RPM are structurally invariant in the sense that the agent uses the same triplet $\langle \succ, \triangleright, \lambda \rangle$ to make a choice from each choice set. As a particular instance of this similarity, both models render a unique representation when there are only three alternatives.[21] As for the general case, our Proposition 1 provides a uniqueness result for the RPM, which can be thought as the counterpart of Falmagne's result for the RUM.

For a given RPM $\langle \succ, \triangleright, \lambda \rangle$, let for each $S \in \Omega$ and $x \in S$, $\lambda(x = B_k|S, \succ)$ be the sum of the weights assigned to the pro-preferences at which x is the kth-best alternative in S. Similarly, let $\lambda(x = W_k|S, \triangleright)$ be the sum of the weights assigned to the con-preferences at which x is the kth-worst alternative in S. In our next result, we show that for each RCF the difference between the the sum of the weights assigned to the pro-preferences at which x is the kth-best alternative in S and the sum of the weights assigned to the con-preferences at which x is the kth-worst alternative in S is the same for each prudential representation of the given RCF. That is, $\lambda(x = B_k|S, \succ) - \lambda(x = W_k|S, \triangleright)$ is fixed for each RPM $\langle \succ, \triangleright, \lambda \rangle$ that represents the given RCF.

Proposition 1 *If $\langle \succ, \triangleright, \lambda \rangle$ and $\langle \succ', \triangleright', \lambda' \rangle$ are random prudential representations of the same RCF p, then for each $S \in \Omega$ and $x \in S$,*

$$\lambda(x = B_k|S, \succ) - \lambda(x = W_k|S, \triangleright) = \lambda'(x = B_k|S, \succ') - \lambda'(x = W_k|S, \triangleright'). \tag{3}$$

Proof. Let $\langle \succ, \triangleright, \lambda \rangle$ and $\langle \succ', \triangleright', \lambda' \rangle$ be two RPMs that represent the same RCF p. Now, for each choice set $S \in \Omega$, both λ and λ' should satisfy the identity (CS) used in Step 1 of the proof of Theorem 1. That is, for each $S \in \Omega$ and $x \in S$ both λ and λ' generates the same $q(x, S)$ value. Therefore, if we can show that $\lambda(x = B_k|S, \succ)$ can be expressed in terms of $q(x, \cdot)$, then (3) follows. To see this, let $\langle \succ, \triangleright, \lambda \rangle$ be any RPM that represents p. Next, for each $S \in \Omega$, $x \in S$, and $k \in \{1, \ldots, |S|\}$, consider a partition (S_1, S_2) of S such that $x \in S_2$ and $|S_1| = k-1$. Let $\mathbb{P}(S, x, k)$ be the collection of all these partitions. Now, for each fixed $(S_1, S_2) \in \mathbb{P}(S, x, k)$, let $\lambda(x|S_1, S_2, \succ)$ be

[21] This directly follows from the construction used to establish the base of induction in Theorem 1's proof.

the sum of the weights of the pro-preferences at which x is the best alternative in S_2 and the worst alternative in S_1. Note that for each such pro-preference, x is the kth-best alternative in S. Similarly, let $\lambda(x|S_1, S_2, \triangleright)$ be the sum of the weights of the con-preferences at which x is the best alternative in S_1 and the worst alternative in S_2. Note that for each such con-preference x is the kth-worst alternative in S. Now, it follows that we have:

$$\lambda(x = B_k|S, \succ) = \sum_{\{(S_1,S_2)\in\mathbb{P}(S,x,k)\}} \lambda(x|S_1, S_2, \succ), \tag{4}$$

$$\lambda(x = W_k|S, \triangleright) = \sum_{\{(S_1,S_2)\in\mathbb{P}(S,x,k)\}} \lambda(x|S_1, S_2, \triangleright). \tag{5}$$

Since for each $T \in \Omega$ such that $S_2 \subset T$ and $T \subset X \setminus S_1$, by definition, $q(x, T)$ gives the difference between the sum of the weights of the pro-preferences at which x is the best alternative in S and sum of the weights of the con-preferences at which x is the worst alternative in S, it follows that

$$\sum_{\mathbb{P}(S,x,k)} \lambda(x|S_1, S_2, \succ) - \sum_{\mathbb{P}(S,x,k)} \lambda(x|S_1, S_2, \triangleright) = \sum_{\mathbb{P}(S,x,k)} \sum_{S_2\subset T\subset X\setminus S_1} q(x, T). \tag{6}$$

Finally, if we substitute (4) and (5) in (6), then we express $\lambda(x = B_k|S, \succ) - \lambda(x = B_k|S, \triangleright)$ only in terms of $q(x, \cdot)$, as desired. ■

2.5 Locally nonnegative Block-Marschak polynomials

As we pointed out previously, a RCF is a RUM if and only if the associated Block-Marschak polynomials are all nonnegative. We explore the implication of Block-Marschak polynomials being only *locally* nonnegative at a given choice set. We argue that by using the construction in Theorem 1's proof, we can specify a prudential representation of the given RCF, which renders a random utility representation for the observed choice behavior at the given choice set. We believe that this observation goes in line with our claim that the prudential model provides a canonical language to study random choice. Next, we define the Block-Marschak polynomials.

Definition 3 *For each RCF p, $S \in \Omega$, and $x \in S$, the associated Block-Marschak polynomial*

$$q(x, S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} p(x, T). \tag{7}$$

Suppose that for a given RCF p, although p is not a RUM, the Block-Marschak polynomials are nonnegative for all the choice sets that contain a given choice set S. That is, for each choice set T with $S \subset T$ and $x \in T$, $q(x, T) \geq 0$. We call these Block-Marschak polynomials the **higher-order Block-Marschak polynomials at** S. In what follows, we argue that we can specify a prudential representation of the given RCF p, which renders a random utility representation for $p(S)$, whenever the higher-order Block-Marschak polynomials at S are nonnegative.

To construct this special prudential representation, suppose that we choose the weights in the $[0, 1]$ interval, whenever (RS) and (CS) values are all nonnegative. First, note that if these weights are nonnegative, then the premises of Lemma 5 are satisfied. Therefore, this specific selection does not create a problem in the remainder of our construction. Next, note that since the higher-order Block-Marschak polynomials at S are nonnegative, the associated (RS) and (CS) values are all nonnegative. Then, the constructed prudential representation $\langle \succ, \triangleright, \lambda \rangle$ boils down to a random utility representation when restricted to the choice set S.

To see how to obtain this random utility representation from the prudential one, suppose that $|S| = n - k$. For each pro-preference $\succ_i$, let $\succ_i^{k+1}$ be the preference obtained by truncating the top $k+1$-ranked alternatives at $\succ_i$. Similarly, for each con-preference $\triangleright_j$, let $\triangleright_j^{k+1}$ be the preference obtained by truncating the bottom $k+1$-ranked alternatives at $\triangleright_j$. Note that since $|S| = n - k$, for each $\succ_i^{k+1}$ and $\triangleright_j^{k+1}$, $max(S, \succ_i^{k+1}) \neq \emptyset$ and $min(S, \triangleright_j^{k+1}) \neq \emptyset$. Now, a truncated–con-preference $\triangleright_j^{k+1}$, when restricted to S, may coincide with the inverse of a truncated–pro-preference $\succ_i^{k+1}$ restricted to S. Define $\lambda|_S$ as the weight function that assigns to each $\succ_i^{k+1}$ the net weight that equals the difference between the weights of such pro- and con-preferences. Note that $\lambda|_S(\succ_i^{k+1})$ is calculated by restricting $\succ_i^{k+1}$ to S, but the weight is assigned to $\succ_i^{k+1}$. It follows from our construction in Theorem 1's proof that if (RS)

and (CS) values are all nonnegative, then $\lambda|_S$ is a probability measure on $\{\succ_i^{k+1}\}_{\succ_i \in \succ}$. Finally, we can extend the truncated pro-preferences to X in any arbitrary way. Thus, $\langle \succ^{k+1}, \lambda|_S \rangle$ yields the desired random utility representation for each choice set T with $S \subset T$.

In a related paper, McClellon (2015) proposes generalizations of the random utility model by restricting the nonnegativity of the higher-order Block-Marschak polynomials. In contrast to the additive representation rendered by the prudential model, the primitives in these models are set functions defined on the space of preferences, which can be nonadditive.

2.6 Prudential choice with respect to a given $(\succ_1, \succ_2)$

In this section, we focus on a particular choice problem in which there are two observable orderings $(\succ_1, \succ_2)$ that are relevant for choice, such as price and quality. This provides a generalization of Example 3, which presents an attraction effect scenario. In our analysis, we provide a set of choice axioms, which guarantee that the observed choices can be generated via an RPM in which the pro-preferences and the con-preferences are obtained from the given preference pair. A notable aspect of this characterization is that attraction effect type of choice behavior is the content of a key axiom (the *attraction* axiom), which indicates the tight connection between the model and the attraction effect phenomena.

Formally, for a given pair of preferences $(\succ_1, \succ_2)$, an RCF p is prudential with respect to $(\succ_1, \succ_2)$, if there exists a weight function λ such that the RPM $\langle \succ, \triangleright, \lambda \rangle$ represents p, where $\succ = (\succ_1, \succ_2)$ and $\triangleright = (\succ_1, \succ_2)$. That is, both the pro-preferences and the con-preferences are the given $(\succ_1, \succ_2)$. For each choice set S and $x \in S$, if $\lambda(x, S) > 0$, then $p(x, S) = \frac{\lambda(x,S)}{\sum_{\{y \in S^+\}} \lambda(y,S)}$; if $\lambda(x, S) \leq 0$, then $p(x, S) = 0$. Next, we provide four axioms and show that the RCFs that are prudential with respect to a given $(\succ_1, \succ_2)$ are the ones that satisfy these axioms.

Our first axiom, *domination*, requires that if an alternative *dominates* another, in the sense that the former is better than the latter in both orderings, then the

dominated one is never chosen when both are available. Formally, for each $x, y \in X$, x **dominates** y, denoted by $x >> y$ if $x \succ_1 y$ and $x \succ_2 y$.

Domination: For each $S \in \Omega$ and $x, y \in S$, if $x >> y$, then $p(y, S) = 0$.

Our second axiom, *attraction*, requires that adding an alternative dominated by another one should not decrease the choice probability of the dominating alternative.

Attraction: For each $S \in \Omega$ and $x, z \in X$, if $x >> z$, then $p(x, S \cup \{z\}) \geq p(x, S)$.

As in an attraction effect scenario, for each $x, y, z \in X$, if neither y dominates x or z, nor x or z dominates y, but x dominates z, then z **is a decoy for** x **when** y **is available.** It directly follows from *attraction* that if z is a decoy for x when y is available, then $p(x, \{x, y, z\}) \geq p(x, \{x, y\})$.

Our third axiom, *best-worst neutrality*, requires that if two choice sets are similar to each other in the sense that the $(\succ_1, \succ_2)$-best alternatives in S can be renamed as to obtain the configuration of the $(\succ_1, \succ_2)$-best alternatives in S' in the best and worst positions, then the choice probabilities should be preserved under this renaming. Formally, a choice set S is **best-worst isomorphic** to another one S', denoted by $S \sim^{\pi} S'$, if there is a one-to-one mapping π between the $(\succ_1, \succ_2)$-best alternatives in S and the $(\succ_1, \succ_2)$-best alternatives S' such that for each $i, j \in \{1, 2\}$ and $x \in max(S, \succ_i)$,

1. $x = max(S, \succ_i)$ if and only if $\pi(x) = max(S', \succ_i)$, and
2. $x = min(S, \succ_j)$ if and only if $\pi(x) = min(S', \succ_j)$.

Best-worst neutrality: For each $S, S' \in \Omega$, if $S \sim^{\pi} S'$, then for each $x \in max(S, \succ_i)$ where $i \in \{1, 2\}$, $p(x, S) = p(\pi(x), S')$.

To introduce our last axiom, we first define the **choice likelihood of** x **from** S as the ratio of the probability that alternative x is chosen from choice set S to the probability that any other alternative is chosen from S, that is, $L(x, S) = \frac{p(x,S)}{1-p(x,S)}$. Next, we present and interpret our last axiom.

Attraction gain equivalence: For each $x, y, z, w \in X$, if z is a decoy for x when y is

available and w is a decoy for y when x is available, then

$$\frac{L(x,\{x,y,z\})}{L(y,\{x,y,w\})} = \frac{L(x,\{x,y,z,w\})}{L(y,\{x,y\})}.$$

To get an intuition for attraction gain independence, note that the two choice likelihood ratios $\frac{L(x,\{x,y,z\})}{L(y,\{x,y,w\})}$ and $\frac{L(x,\{x,y,z,w\})}{L(y,\{x,y\})}$ can be interpreted as measuring the attraction gain of x relative to that of y. In that, former is the ratio of the choice likelihood of x to y when each alternative's decoy is added separately. The latter is the ratio of the choice likelihood of x in the availability of both decoys, to the choice likelihood of y in the absence of any decoy. Attraction gain equivalence requires these two plausible measures of relative attraction gain be equal. Next we state our characterization result. We present the proof in Section 6.

Proposition 2 *For a given $(\succ_1, \succ_2)$, an RCF p is prudential w.r.t. $(\succ_1, \succ_2)$ if and only if p satisfies domination, attraction, best-worst neutrality, and attraction gain equivalence.*

We assume that $(\succ_1, \succ_2)$ are given. One follow-up question is whether we can identify prudential choice by deriving $(\succ_1, \succ_2)$ from agent's choices. In this vein, Eliaz et al. (2011) provide an axiomatic characterization of the *top-and-top choice rule*, which chooses the $(\succ_1, \succ_2)$-best alternatives for a pair of preferences $(\succ_1, \succ_2)$ obtained from an agent's deterministic choices. We conjecture that the above four axioms, together with Eliaz et al.'s axioms provide a characterization of the RCFs that render a prudential representation with respect to an unobserved preference pair. A caveat is that our axioms refer to the observed preferences through the *domination relation* that we have defined. To overcome this difficulty, we propose to replace the existing *domination* relation with the following commonly used one: an alternative x *dominates** another alternative y if y is never chosen when x is available. Then, all our axioms are well defined with unobserved preferences.

The specific prudential model analyzed in this section is related to the literature on dual-self models. Among these, **?** propose a deterministic choice model in which an agent seeks to reach a compromise between two inner selves that represent two attributes of the available alternatives. As we do in this section, they also assume that

two orderings that represent the two inner selves are observable. They characterize a model of reason-based choice obtained as a result of a cooperative solution to the bargaining problem between the two selves, which accounts for both the deterministic formulations of the attraction and the comprimise effects. Another related paper is Manzini & Mariotti (2017), who provide a characterization of the random utility model with two preferences. In here, we assume that the preferences are observed, and used not only on the pro-side, but also on the con-side. Therefore, we end up in substantially different characterizations.

3 Prudential deterministic choice functions

3.1 The model

A (deterministic) choice function C is a mapping that assigns each choice set $S \in \Omega$ a member of S, that is $C : \Omega \to X$ such that $C(S) \in S$. Let $\succ$ and $\triangleright$ stand for two collections of preferences on X as before. A **(deterministic) prudential model (PM)** is a pair $\langle \succ, \triangleright \rangle$ consisting of the pro-preferences and the con-preferences. As before, define $Pros(x, S) = \{\succ_i \in \succ : x = max(S, \succ_i)\}$ and $Cons(x, S) = \{\triangleright_i \in \triangleright : x = min(S, \triangleright_i)\}$.

Definition 4 *A choice function C is **prudential** if there is an PM $\langle \succ, \triangleright \rangle$ such that for each choice set $S \in \Omega$ and $x \in S$, $C(S) = x$ if and only if $|Pros(x, S)| > |Cons(x, S)|$.*

Note that our prudential model is not a direct adaptation of its random counterpart. In that, we require each preference to have a fixed unit weight, instead of having fractional weights. Moreover, if an agent is prudential, then at each choice set S there should be a single alternative x such that the number of $Pros(x, S)$ is greater than the number of $Cons(x, S)$. Therefore, each prudential model $\langle \succ, \triangleright \rangle$ may not render a well-defined choice function. However, as in the random setup, one can consider a less structured formulation that always renders a choice function. For this alternative formulation, suppose that for each given prudential model $\langle \succ, \triangleright \rangle$, and

choice set S, first, the alternatives with the maximum $Pros(x, S) - Cons(x, S)$ value are shortlisted, then one of them is singled out. It directly follows that if a choice function C renders a prudential representation as specified in Definition 4, then C can also be represented in this alternative form. However, showing the converse is a nontrivial exercise. Next, to illustrate how the model works, we revisit Luce and Raiffa's dinner example (Luce & Raiffa (1957)) by following a prudential model.

Example 4 Suppose you choose chicken when the menu consists of steak and chicken only, yet go for the steak when the menu consists of steak (S), chicken (C), and fish (F). Consider the pro-preferences $\succ_1$ and $\succ_2$ that order the three dishes according to their *attractiveness* and *healthiness*, so suppose $S \succ_1 F \succ_1 C$ and $C \succ_2 F \succ_2 S$. As a con-preference, consider $C \triangleright S \triangleright F$, which orders the dishes according to their *riskiness*. Since cooking fish requires expertise, it is the most risky one, and since chicken is the safest option, it is the least risky one. Now, to make a choice from the grand menu, the pros are: "S is the most attractive", "F is the most healthy", but also "F is the most risky". Thus, S is chosen from the grand menu. If only S and C are available, then we have "C is the most healthy", "S is the most attractive", but also "S is the most risky", so C is chosen.

In our Definition 4, although we do not restrict the structure of the pro- and con-preferences, we ask for a rather structured representation in the vein of Franklin's prudential algebra. We see at least two benefits of the stringency. First, we obtain the uniqueness property presented in Section 3.3. Second, given our Theorem 2, one can use our representation to identify other inclusive choice models, which otherwise may not be an easy exercise. In Section 3.4, we present an application along these lines, in which we show that each choice function is plurality-rationalizable.

3.2 Main result

By using the construction in the proof of Theorem 1 and two well-known results from integer-programming literature, we show that every choice function is prudential.

This result does not directly follow from Theorem 1, since the prudential model is not a direct adaptation of its random counterpart. Next, we present the result and its proof.

Theorem 2 *Every choice function is prudential.*

Proof. We prove this result by following the construction used to prove Theorem 1. So, we proceed by induction. Note that since C is a deterministic choice function, for each $x_i \in X$, $\lambda^1([\succ^{x_i}]) \in \{0, 1\}$. Next, by proceeding inductively, we assume that for any $k \in \{1, \ldots, n-1\}$, there is a signed weight function λ^k that takes values $\{-1, 0, 1\}$ over $\mathcal{P}^k$ and represents C_k. It remains to show that we can construct λ^{k+1} taking values $\{-1, 0, 1\}$ over $\mathcal{P}^{k+1}$, and that represents C_{k+1}. We know from Step 1 of the proof of Theorem 1 that to show this it is sufficient to construct λ^{k+1} such that (RS) and (CS) holds. However, this time, in addition to satisfying (RS) and (CS), we require each $\lambda^{k+1}_{ij} \in \{-1, 0, 1\}$.

First, note that equalities (RS) and (CS) can be written as a system of linear equations: $A\lambda = b$, where $A = [a_{ij}]$ is a $(k! + (n-k)) \times (n-k)k!$ matrix with entries $a_{ij} \in \{0, 1\}$, and $b = [\lambda^k([\succ^k_1]), \ldots, \lambda^k([\succ^k_{k!}]), q(x_1, S), \ldots, q(x_{n-k}, S)]$ is the column vector of size $k! + (n-k)$. Let Q denote the associated polyhedron, i.e. $Q = \{\lambda \in \mathbb{R}^{(n-k)k!} : A\lambda = b \text{ and } -1 \leq \lambda \leq 1\}$. A matrix is **totally unimodular** if the determinant of each square submatrix is 0, 1 or -1. Following result directly follows from Theorem 2 of Hoffman & Kruskal (2010).

Lemma 1 (Hoffman & Kruskal (2010)) *If matrix A is totally unimodular, then the vertices of Q are integer valued.*

Heller & Tompkins (1956) provide the following sufficient condition for a matrix being totally unimodular.

Lemma 2 (Heller & Tompkins (1956)) *Let A be an $m \times n$ matrix whose rows can be partitioned into two disjoint sets R_1 and R_2. Then, A is totally unimodular if:*

1. *Each entry in A is 0, 1, or -1;*

2. *Each column of A contains at most two non-zero entries;*

3. *If two non-zero entries in a column of A have the same sign, then the row of one is in R_1, and the other is in R_2;*

4. *If two non-zero entries in a column of A have opposite signs, then the rows of both are in R_1, or both in R_2.*

Next, by using Lemma 2, we show that the matrix that is used to define (RS) and (CS) as a system of linear equations is totally unimodular. To see this, let A be the matrix defining the polyhedron Q. Since $A = [a_{ij}]$ is a matrix with entries $a_{ij} \in \{0, 1\}$, (1) and (4) are directly satisfied. To see that (2) and (3) also hold, let $R_1 = [1, \ldots, k!]$ consist of the the first $k!$ rows and $R_2 = [1, \ldots, n-k]$ consist of the the remaining $n-k$ rows of A. Note that for each $i \in R_1$, the ith row A_i is such that $A_i\lambda = \lambda^k([\succ_i^k])$. That is, for each $j \in \{(i-1)k!, \ldots, ik!\}$, $a_{ij} = 1$ and the rest of A_i equals 0. For each $i \in R_2$, the ith row A_i is such that $A_i\lambda = q(x_i, A)$. That is, for each $j \in \{i, i+k!, \ldots, i+(n-k-1)k!\}$, $a_{ij} = 1$ and the rest of A_i equals 0. To see that (2) and (3) hold, note that for each $i, i' \in R_1$ and $i, i' \in R_2$, the non-zero entries of A_i and $A_{i'}$ are disjoint. It follows that for each column there can be at most two rows with value 1, one in R_1 and the other in R_2.

Finally, it follows from the construction in Step 3 of the proof of Theorem 1 that Q is nonempty, since there is λ vector with entries taking values in the $[-1, 1]$ interval. Since, as shown above, A is totally unimodular, it directly follows from Lemma 1 that the vertices of Q are integer valued. Therefore, λ^{k+1} can be constructed such that (RS) and (CS) holds, and each $\lambda_{ij}^{k+1} \in \{-1, 0, 1\}$. ■

3.3 Uniqueness

For a given PM $\langle \succ, \triangleright \rangle$, let for each $S \in \Omega$ and $x \in S$, $Pros^k(x, S)$ be the set of pro-preferences at which x is the kth-best alternative in S. Similarly, let $Cons^k(x, S)$ be the set of con-preferences at which x is the kth-worst alternative in S. In our next result, we show that for a given choice function, the difference between the the

number of pro-preferences at which x is the kth-best alternative in S and the number of con-preferences at which x is the kth-worst alternative in S is the same for each prudential representation of the given choice function. We obtain this result as a direct corollary to our Proposition 1.

Corollary 1 *If $\langle \succ, \triangleright \rangle$ and $\langle \succ', \triangleright' \rangle$ are prudential representations of the same choice function C, then for each $S \in \Omega$, $x \in S$, and $k \in \{1, \ldots, n\}$, both representations lead the same $|Pros^k(x, S)| - |Cons^k(x, S)|$ value.*

Proof. Since each pro- and con-preference has a unit weight at each prudential representation of a given choice function, $|Pros^k(x, S)| - |Cons^k(x, S)|$ equals $\lambda(x = B_k|S, \succ) - \lambda(x = W_k|S, \triangleright)$. Then, it follows from Proposition 1 that $|Pros^k(x, S)| - |Cons^k(x, S)|$ is fixed for each prudential representation. ■

3.4 Plurality-rationalizable choice functions

We propose a collective decision making model based on plurality voting. It turns out that this model is closely related to our prudential choice model. To introduce this model, let $\succ^* = [\succ_1^*, \ldots, \succ_m^*]$ be a preference profile, which is a list of preferences. In contrast to a collection of preferences, denoted by $\succ$, a preference $\succ_i$ can appear more than once in a preference profile $\succ^*$. For each choice set $S \in \Omega$ and $x \in S$, x is the **plurality winner of $\succ^*$ in** S if for each $y \in S \setminus \{x\}$, the number of preferences in $\succ^*$ that top ranks x in S is more than the number of preferences in $\succ^*$ that top ranks y in S. That is, for each $y \in S \setminus \{x\}$, $|\{\succ_i^* : \ x = max(S, \succ_i^*)\}| > |\{\succ_i^* : \ y = max(S, \succ_i^*)\}|$. Next, we define plurality-rationalizability, then by using our Theorem 2, we show that every choice function is plurality-rationalizable.

Definition 5 *A choice function C is **plurality-rationalizable** if there is preference profile $\succ^*$ such that for each choice set $S \in \Omega$ and $x \in S$, $C(S) = x$ if and only if x is the plurality winner of $\succ^*$ in S.*

Proposition 3 *Every choice function is plurality-rationalizable.*

Proof. Let C be a choice function. It follows from Theorem 2 that C is prudential. Let the PM $\langle \succ, \triangleright \rangle$ be such that for each choice set $S \in \Omega$ and $x \in S$, $C(S) = x$ if and only if $|Pros(x, S)| > |Cons(x, S)|$. Now, to construct the desired preference profile, first consider the list of all preferences defined on X. Then, eliminate any preference that belongs to $\triangleright$ and add any preference that belongs to $\succ$. Let $\succ^*$ be the obtained preference profile. Next, consider a choice set $S \in \Omega$ and suppose $C(S) = x$. In what follows we show that x is the plurality winner of $\succ^*$ in S. We know that $|Pros(x, S)| > |Cons(x, S)|$ and for each $y \in S \setminus \{x\}$, $|Pros(y, S)| \leq |Cons(y, S)|$. It follows that for each $y \in S \setminus \{x\}$, $|Pros(x, S)| - |Cons(x, S)| > |Pros(y, S)| - |Cons(y, S)|$. Now, note that by construction of $\succ^*$, for each $y \in S$ the number of preferences in $\succ^*$ that top ranks y in S equals the number of all preferences that top ranks y in S added to $|Pros(y, S)| - |Cons(y, S)|$. Since for each $y \in S$, the number of all preferences that top ranks y in S is fixed, it follows that x is the plurality winner of $\succ^*$ in S. ■

Remark 3 One can consider an even more stringent model, in which we require that an alternative x is chosen from a choice set S *at the margin*, in the sense that $x = C(S)$ if and only if for each $y \in S \setminus \{x\}$, $|\{\succ_i^* : \ x = max(S, \succ_i^*)\}| - |\{\succ_i^* : \ y = max(S, \succ_i^*)\}| = 1$. We obtain the same anything–goes-result with this more demanding model by following the proof of Proposition 3.

In an early paper McGarvey (1953) shows that for each asymmetric and complete binary relation, there exists a preference profile such that the given binary relation is obtained from the preference profile by comparing each pair of alternatives via majority voting. For antisymmetric and complete binary relations (without indifferences), we obtain McGarvey's result, as a corollary to Proposition 3. To see this, note that if we restrict a choice function to binary choice sets, then we obtain an antisymmetric and complete binary relation. Since for binary choices, being a plurality winner means being a majority winner, McGarvey's result directly follows.

4 Conclusion

As we have shown, prudential model provides a unified lens through which any deterministic or stochastic choice behavior can be explained. The structural invariance of the prudential model reflects itself as a form of uniqueness, which facilitates identifying the model's primitives from observed choices. We hope that the analytic results that we obtain here would stimulate some empirical work.

In our examples we present specific prudential models that accommodate commonly observed choice including attraction effect and similarity effect by capturing the key aspects of the contexts in which these choice patterns are observed. These examples indicate that analyzing prudential choice model with restricted pro- and con-preferences may lead to insightful results. In this vein, we analyzed choice problems in which there are two observable orderings of the alternatives that are relevant for choice. Knowing that each choice function is prudential facilitates identification of other inclusive choice models. We present an application along these lines, in which we show that each choice function is plurality-rationalizable. Although our study covers a rather extensive treatment of the prudential model, we can hardly claim that it is exhaustive, as it leads to a wide variety of directions yet to be pursued.

5 Proof of Theorem 1

We start by proving some lemmas that are critical for proving the theorem. First, we use a result by Ford Jr & Fulkerson (2015)[22] as Lemma 3. Then, our Lemma 4 follows directly. Next, by using Lemma 4, we prove Lemma 5, which shows that, under suitable conditions, Lemma 3 holds for any real-valued row and column vectors.

Lemma 3 (Ford Jr & Fulkerson (2015)) *Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be positive real-valued vectors with $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. There is an $m \times n$ matrix $A = [a_{ij}]$ such that A has row sum vector R and column sum vector C, and each entry $a_{ij} \in [0, 1]$ if and only if for each $I \subset \{1, 2, \ldots, m\}$ and $J \subset \{1, 2, \ldots, n\}$,*

$$|I||J| \geq \sum_{i \in I} r_i - \sum_{j \notin J} c_j. \tag{FF}$$

Lemma 4 *Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be positive real-valued vectors with $0 \leq r_i \leq 1$ and $0 \leq c_j \leq m$ such that $\sum_{i=1}^m r_i = \sum_{j=1}^n c_i$. Then there is an $m \times n$ matrix $A = [a_{ij}]$ such that A has row sum vector R and column sum vector C, and each entry $a_{ij} \in [0, 1]$.*

Proof. Given such R and C, since for each $i \in \{1, 2, \ldots, m\}$, $0 \leq r_i \leq 1$, we have for each $I \subset \{1, 2, \ldots, m\}$, $\sum_{i \in I} r_i \leq |I|$. Then, it directly follows that (FF) holds. ■

Next by using Lemma 4, we prove Lemma 5, which plays a key role in proving Theorem 1.

Lemma 5 *Let $R = [r_1, \ldots, r_m]$ and $C = [c_1, \ldots, c_n]$ be real-valued vectors with $-1 \leq r_i \leq 1$ and $-m \leq c_j \leq m$ such that $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. If $2m \geq \sum_{i=1}^m |r_i| + \sum_{j=1}^n |c_j|$, then there is an $m \times n$ matrix $A = [a_{ij}]$ such that:*

i. A has row sum vector R and column sum vector C,

[22] This result, as stated in Lemma 3, but with integrality assumptions on R, C, and A follows from Theorem 1.4.2 in Brualdi & Ryser (1991), and they report that Ford Jr & Fulkerson (2015) proves, by using network flow techniques, that the theorem remains true if the integrality assumptions are dropped and the conclusion asserts the existence of a real nonnegative matrix.

ii. each entry $a_{ij} \in [-1, 1]$*, and*

iii. for each $j \in \{1, \ldots, n\}$*,* $\sum_{i=1}^{m} |a_{ij}| \leq |c_j| + \max\{0, \frac{\sum_{i=1}^{m} |r_i| - \sum_{j=1}^{n} |c_j|}{n}\}$.

Proof. Since r_i and c_j values can be positive or negative, although the sum of the rows equals the sum of the column, their absolute values may not be the same. We analyze two cases separately, where $\sum_{i=1}^{m} |r_i| \geq \sum_{j=1}^{n} |c_j|$ and $\sum_{i=1}^{m} |r_i| < \sum_{j=1}^{n} |c_j|$. Before proceeding with these cases, first we introduce some notation and make some elementary observations.

For each real number x, let $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$. Note that for each x, $x^+ + x^- = x$. Let $R^+ = [r_1^+, \ldots, r_m^+]$ and $R^- = [r_1^-, \ldots, r_m^-]$. Define the n-vectors C^+ and C^- respectively. Next, let $\Sigma_{R^+} = \sum_{i=1}^{m} r_i^+$, $\Sigma_{R^-} = \sum_{i=1}^{m} r_i^-$, $\Sigma_{C^+} = \sum_{j=1}^{n} c_j^+$ and $\Sigma_{C^-} = \sum_{j=1}^{n} c_j^-$. That is, $\Sigma_{R^+}(\Sigma_{R^-})$ and $\Sigma_{C^+}(\Sigma_{C^-})$ are the sum of the positive (negative) rows in R and columns in C. Since the sum of the rows equals the sum of the columns, we have $\Sigma_{R^+} + \Sigma_{R^-} = \Sigma_{C^+} + \Sigma_{C^-}$.

For each row vector R and column vector C, suppose for each $i \in \{1, \ldots, m_1\}$, $r_i \geq 0$ and for each $i \in \{m_1 + 1, \ldots, m\}$, $r_i < 0$. Similarly, suppose for each $j \in \{1, \ldots, n_1\}$, $c_j \geq 0$ and for each $j \in \{n_1 + 1, \ldots, n\}$, $c_j < 0$. Now, let $R^1(R^2)$ be the m_1-vector ($(m - m_1)$-vector), consisting of the non-negative (negative) components of R. Similarly, for each column vector C, let $C^1(C^2)$ be the n_1-vector ($(n - n_1)$-vector), consisting of the non-negative (negative) components of C. It directly follows from the definitions that $\sum_{i=1}^{m_1} r_i = \sum_{i=1}^{m} r_i^+$ and $\sum_{i=m_1+1}^{m} r_i = \sum_{i=1}^{m} r_i^-$. Similarly, $\sum_{j=1}^{n_1} c_j = \sum_{j=1}^{n} c_j^+$ and $\sum_{j=n_1+1}^{n} c_j = \sum_{j=1}^{n} c_j^-$.

Case 1: Suppose that $\sum_{i=I} |r_i| \geq \sum_{j \in J} |c_j|$. First, for each $j \in \{1, \ldots, n\}$, let

$$\epsilon_j = \frac{\Sigma_{R^+} - \Sigma_{C^+}}{n}.$$

Note that since $\sum_{i=1}^{m} |r_i| \geq \sum_{j=1}^{n} |c_j|$, we have $\Sigma_{R^+} \geq \Sigma_{C^+}$ and $\Sigma_{R^-} \leq \Sigma_{C^-}$. Moreover, since the sum of the rows equals the sum of the columns, we have $\Sigma_{R^+} - \Sigma_{C^+} = \Sigma_{C^-} - \Sigma_{R^-}$. Therefore, by the choice of ϵ_j, we get

$$\sum_{i=1}^{m} r_i^+ = \sum_{j=1}^{n} c_j^+ + \epsilon_j \text{ and } \sum_{i=1}^{m} r_i^- = \sum_{j=1}^{n} c_j^- - \epsilon_j. \tag{8}$$

Next, consider row-column vector pairs $(R^1, C^+ + \epsilon)$ and $(-R^2, -(C^- - \epsilon))$, where ϵ is the non-negative n-vector such that each ϵ_j is as defined above. It follows from (8) that for both pairs the sum of the rows equals the sum of the columns. Now we apply Lemma 4 to the row-column vector pairs $(R^1, C^+ + \epsilon)$ and $(-R^2, -(C^- - \epsilon))$. It directly follows that there exists a positive $m_1 \times n$ matrix A^+ and a negative $(m - m_1) \times n$ matrix A^- that satisfy (i) and (ii). We will obtain the desired matrix A by augmenting A^+ and A^-. We illustrate A^+ and A^- below.

	$(c_1^+ + \epsilon_1)$ $(c_2^+ + \epsilon_2)$ $(c_3^+ + \epsilon_3)$ $\cdots$ $(c_n^+ + \epsilon_n)$	
$r_1 \geq 0$ $r_2 \geq 0$ $\vdots$ $r_{m_1} \geq 0$	A^+	
	A^-	$r_{m_1+1} < 0$ $\vdots$ $r_m < 0$
	$(c_1^- - \epsilon_1)$ $(c_2^- - \epsilon_2)$ $(c_3^- - \epsilon_3)$ $\cdots$ $(c_n^- - \epsilon_n)$	

Since A^+ and A^- satisfy (i) and (ii), A satisfies (i) and (ii). To see that A satisfies (iii), for each $j \in \{1, \ldots, n\}$, consider $\sum_{i=1}^m |a_{ij}|$. Note that, by the construction of A^+ and A^-, for each $j \in \{1, \ldots, n\}$,

$$\sum_{i=1}^m |a_{ij}| = c_j^+ + \epsilon_j + (-c_j^- + \epsilon_j) = |c_j| + 2\epsilon_j = |c_j| + 2\frac{\Sigma_{R^+} - \Sigma_{C^+}}{n}. \tag{9}$$

Since for each $j \in \{1, \ldots, n\}$, $c_j = c_j^+ + c_j^-$ such that either $c^+ = 0$ or $c_j^- = 0$, we get $|c_j| = c_j^+ - c_j^-$. To see that (iii) holds, observe that $\sum_{i=1}^m |r_i| - \sum_{j=1}^n |c_j| = \Sigma_{R^+} - \Sigma_{C^+} + \Sigma_{C^-} - \Sigma_{R^-}$. Since the sum of the rows equals the sum of the columns, i.e. $\Sigma_{R^+} + \Sigma_{R^-} = \Sigma_{C^+} + \Sigma_{C^-}$, we also have $\Sigma_{R^+} - \Sigma_{C^+} = \Sigma_{C^-} - \Sigma_{R^-}$. This observation, together with (9), implies that (iii) holds.

Case 2 Suppose that $\sum_{i=1}^m |r_i| < \sum_{j=1}^n |c_j|$. First, we show that there exists a non-negative m-vector ϵ such that

(E1) for each $i \in \{1, \ldots, m\}$, $r_i^+ + \epsilon_i \leq 1$ and $r_i^- - \epsilon_i \geq -1$, and

(E2) $\sum_{i=1}^{m} r_i^+ + \epsilon_i = \sum_{j=1}^{n} c_j^+$ (equivalently $\sum_{i=1}^{m} r_i^- - \epsilon_i = \sum_{j=1}^{n} c_j^-$) holds.

Step 1: We show that if $\Sigma_{C^+} - \Sigma_{R^+} \leq m - \sum_{i=1}^{m} |r_i|$, then there exists a non-negative m-vector ϵ that satisfies (E1) and (E2). To see this, first note that $m - \sum_{i=1}^{m} |r_i| = \sum_{i=1}^{m}(1 - |r_i|)$. Next, note that, by simply rearranging the terms, we can rewrite (E2) as follows:

$$\sum_{i=1}^{m} \epsilon_i = \Sigma_{C^+} - \Sigma_{R^+} \,. \tag{10}$$

Since $\Sigma_{C^+} - \Sigma_{R^+} \leq \sum_{i=1}^{m}(1 - |r_i|)$, for each $i \in \{1, \ldots, m\}$, we can choose an ϵ_i such that $0 \leq \epsilon_i \leq 1 - |r_i|$ and (10) holds. It directly follows that the associated ϵ vector satisfies (E1) and (E2).

Step 2: We show that since $2m \geq \sum_{i=1}^{m} |r_i| + \sum_{j=1}^{n} |c_j|$, we have $\Sigma_{C^+} - \Sigma_{R^+} \leq m - \sum_{i=1}^{m} |r_i|$. First, it directly follows from the definitions that

$$\sum_{i=1}^{m} |r_i| + \sum_{j=1}^{n} |c_j| = \Sigma_{R^+} - \Sigma_{R^-} + \Sigma_{C^+} - \Sigma_{C^-} \,.$$

Since the sum of the rows equals the sum of the columns, i.e. $\Sigma_{R^+} + \Sigma_{R^-} = \Sigma_{C^+} + \Sigma_{C^-}$, we also have $\Sigma_{R^+} - \Sigma_{C^-} = \Sigma_{C^+} - \Sigma_{R^-}$. It follows that

$$\Sigma_{C^+} - \Sigma_{R^-} \leq m.$$

Finally, if we subtract $\sum_{i=1}^{m} |r_i|$ from both sides of this equality, we obtain $\Sigma_{C^+} - \Sigma_{R^+} \leq m - \sum_{i=1}^{m} |r_i|$, as desired.

It follows from Step 1 and Step 2 that there exists a non-negative m-vector ϵ that satisfies (E1) and (E2). Now, consider the row-column vector pairs $(R^+ + \epsilon, C^1)$ and $(-(R^- - \epsilon), -C^2)$. Since ϵ satisfies (E1) for each $i \in \{1, \ldots, m\}$, $r_i^+ + \epsilon_i \in [0, 1]$ and $r_i^- - \epsilon_i \in [-1, 0]$. Since ϵ satisfies (E2), for both of the row-column vector pairs the sum of the rows equals the sum of the columns. Therefore, we can apply Lemma 4 to row-column vector pairs $(R^+ + \epsilon, C^1)$ and $(-(R^- - \epsilon), -C^2)$. It directly follows that there exists a positive $m \times n_1$ matrix A^+ and a negative $m \times (n - n_1)$ matrix A^- that satisfy (i) and (ii). We obtain the desired matrix A by augmenting A^+ and A^-. We illustrate A^+ and A^- below.

	$c_1 \quad c_2 \quad \cdots \quad c_{n_1} \geq 0$		
$(r_1^+ + \epsilon_1)$			$(r_1^- - \epsilon_1)$
$(r_2^+ + \epsilon_2)$	A^+	A^-	$(r_2^- - \epsilon_2)$
$\vdots$			$\vdots$
$\vdots$			$\vdots$
$(r_m^+ + \epsilon_m)$			$(r_m^- - \epsilon_m)$
		$c_{n_1+1} < 0 \quad \cdots \quad c_n$	

Since A^+ and A^- satisfy (i) and (ii), A satisfies (i) and (ii). In this case, since we did not add anything to the columns and each entry in $A^+(A^-)$ is non-negative (negative), for each $j \in \{1, \ldots, n\}$, $\sum_{i=1}^m |a_{ij}| = |c_j|$. Therefore, A also satisfies (iii). ■

To prove Theorem 1, let p be an RCF and $\mathcal{P}$ denote the collection of all preferences on X. First, we show that there is a **signed weight function** $\lambda : \mathcal{P} \to [-1, 1]$ that **represents** p, i.e. for each $S \in \Omega$ and $x \in S$, $p(x, S)$ is the sum of the weights over $\{\succ_i \in \mathcal{P} \ : \ x = max(S, \succ_i)\}$. Note that λ can assign negative weights to preferences. Once we obtain this signed weight function λ, let $\succ$ be the collection of preferences that receive positive weights, and let $\triangleright'$ be the collection of preferences that receive negative weights. Let $\triangleright$ be the collection of the inverse of the preferences in $\triangleright'$. Finally, let λ^* be the weight function obtained from λ by assigning the absolute value of the weights assigned by λ. It directly follows that p is prudential with respect to the RPM $\langle \succ, \triangleright, \lambda^* \rangle$. We first introduce some notation and present crucial observations to construct the desired signed weight function λ.

Let p be a given RCF and Let $q : X \times \Omega \to \mathbb{R}$ be a mapping such that for each $S \in \Omega$ and $a \notin S$, $q(a, S) = q(a, S \cup \{a\})$ holds. Next, we present a result that is directly obtained by applying the *Möbius inversion*.[23]

[23] See Stanley (1997), Section 3.7. See also Fiorini (2004), who makes the same observation.

Lemma 6 *For each choice set $S \in \Omega$, and alternative $a \in S$,*

$$p(a, S) = \sum_{S \subset T \subset X} q(a, T) \tag{11}$$

if and only if

$$q(a, S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} p(a, T) \tag{12}$$

Proof. For each alternative $a \in X$, note that $p(a, \cdot)$ and $q(a, \cdot)$ are real-valued functions defined on the domain consisting of all $S \in \Omega$ with $a \in S$. Then, by applying the Möbius inversion, we get the conclusion. ■

Lemma 7 *For each choice set $S \in \Omega$ with $|S| = n - k$,*

$$\sum_{a \in X} |q(a, S)| \leq 2^k. \tag{13}$$

Proof. First, note that (13) can be written as follows:

$$\sum_{a \in S} |q(a, S)| + \sum_{b \notin S} |-q(b, S)| \leq 2^k. \tag{14}$$

For a set of real numbers,$\{x_1, x_2, \ldots x_n\}$, to show $\sum_{i=1}^{n} |x_i| \leq 2d$, it suffices to show that for each $I \subset \{1, 2, \cdots, n\}$, we have $-d \leq \sum_{i \in I} x_i \leq d$. Now, as the set of real numbers, consider $\{q(a, S)\}_{a \in X}$. It follows that to show that (14) holds, it suffices to show that for each $S_1 \subset S$ and $S_2 \subset X \setminus S$,

$$-2^{k-1} \leq \sum_{a \in S_1} q(a, S) - \sum_{b \in S_2} q(b, S) \leq 2^{k-1}$$

holds. To see this, first, for each $S_1 \subset S$ and $S_2 \subset X \setminus S$, it follows from Lemma 6 that for each $a \in S_1$ and for each $b \in S_2$, we have

$$q(a, S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} p(a, T) \text{ and } q(b, S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|-1} p(b, T). \tag{15}$$

Note that we obtain the second equality from Lemma 6, since for each $b \notin S$, by definition of $q(b, S)$, we have $q(b, S) = q(b, S \cup \{b\})$. Next, note that for each $T \in \Omega$

with $S \subset T$, $a \in S$, and $b \notin S$, $p(a,T)$ has the opposite sign of $p(b,T)$. Now, suppose for each $b \in S_2$, we multiply $q(b,S)$ with -1. Then, it follows from (15) that

$$\sum_{a \in S_1} q(a,S) - \sum_{b \in S_2} q(b,S) = \sum_{S \subset T \subset X} (-1)^{|T|-|S|} \sum_{a \in S_1 \cup S_2} p(a,T). \tag{16}$$

Note that, for each $T \in \Omega$ such that $S \subset T$, $\sum_{a \in S_1 \cup S_2} p(a,T) \in [0,1]$. Therefore, the term $(-1)^{|T|-|S|} \sum_{a \in S_1 \cup S_2} p(a,T)$ adds at most 1 to the right-hand side of (16) if $|T|-|S|$ is even, and at least -1 if $|T|-|S|$ is odd. Since $|S| = n-k$, for each m with $n-k \leq m \leq n$, there are $\binom{k}{m-n+k}$ possible choice sets $T \in \Omega$ such that $S \subset T$ and $|T| = m$. Moreover, for each $i \in \{1,\ldots,k\}$, there are $\binom{k}{i}$ possible choice sets T such that $S \subset T$ and $|T| = n-k+i$. Now, the right-hand side of (16) reaches its maximum (minimum) when the negative (positive) terms are 0 and the positive (negative) terms are $1(-1)$. Thus, we get

$$-\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2i+1} \leq \sum_{S \subset T \subset X} (-1)^{|T|-|S|} \sum_{a \in S_1 \cup S_2} p(a,T) \leq \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i}.$$

It follows from the *binomial theorem* that both leftmost and rightmost sums are equal to 2^{k-1}. This, combined with (16), implies

$$-2^{k-1} \leq \sum_{a \in S_1} q(a,S) - \sum_{b \in S_2} q(b,S) \leq 2^{k-1}.$$

Then, as argued before, it follows that $\sum_{a \in X} |q(a,S)| \leq 2^k$. ■

Now, we are ready to complete the proof of Theorem 1. Recall that we assume $|X| = n$. For each $k \in \{1,\ldots,n\}$, let $\Omega_k = \{S \in \Omega \ : \ |S| > n-k\}$. Note that $\Omega_n = \Omega$ and $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n$. For each pair of preferences $\succ_1, \succ_2 \in \mathcal{P}$, $\succ_1$ is k-**identical** to $\succ_2$, denoted by $\succ_1 \sim_k \succ_2$, if the first k-ranked alternatives are the same. Note that $\sim_k$ is an equivalence relation on $\mathcal{P}$. Let $\mathcal{P}^k$ be the collection of preferences, such that each set (equivalence class) contains preferences that are k-identical to each other ($\mathcal{P}^k$ is the quotient space induced from $\sim_k$). For each $k \in \{1,\ldots,n\}$, let $[\succ^k]$ denote an **equivalence class** at $\mathcal{P}^k$, where $\succ^k$ linearly orders a fixed set of k alternatives in X.

Note that for each $k \in \{1,\ldots,n\}$, $S \in \Omega_k$ and $\succ_1, \succ_2 \in \mathcal{P}$, if $\succ_1 \sim_k \succ_2$, then since S contains more than $n-k$ alternatives, $\max(\succ_1, S) = \max(\succ_2, S)$. Therefore,

for each $S \in \Omega_k$, it is sufficient to specify the weights on the equivalence classes contained in $\mathcal{P}^k$ instead of all the weights over $\mathcal{P}$. Let p_k be the restriction of p to Ω_k. Similarly, if λ is a signed weight function over $\mathcal{P}$, then let λ^k be the restriction of λ to $\mathcal{P}^k$, i.e. for each $[\succ^k] \in \mathcal{P}^k$, $\lambda^k[\succ^k] = \sum_{\succ_i \in [\succ^k]} \lambda(\succ_i)$. It directly follows that λ represents p if and only if for each $k \in \{1, \ldots, n\}$, λ^k represents p_k. In what follows, we inductively show that for each $k \in \{1, \ldots, n\}$, there is a signed weight function λ^k over $\mathcal{P}^k$ that represents p_k. For $k = n$ we obtain the desired λ.

For $k = 1$, $\Omega_1 = \{X\}$ and $\mathcal{P}^1$ consists of n-many equivalence classes such that each class contains all the preferences that top rank the same alternative, irrespective of whether these are chosen with positive probability. That is, if $X = \{x_1, \ldots, x_n\}$, then $\mathcal{P}^1 = \{[\succ^{x_1}], \cdots, [\succ^{x_n}]\}$, where for each $i \in \{1, \ldots, n\}$ and $\succ_i \in [\succ^{x_i}]$, $\max(X, \succ_i) = x_i$. Now, for each $x_i \in X$, define $\lambda^1([\succ^{x_i}]) = p(x_i, X)$. It directly follows that λ^1 is a signed weight function over $\mathcal{P}^1$ that represents p_1.

For $k = 2$, $\Omega_2 = \{X\} \cup \{X \setminus \{x\}\}_{x \in X}$ and $\mathcal{P}^2$ consists of $\binom{n}{2}$-many equivalence classes such that each class contains all the preferences that top rank the same two alternatives. Now, for each $[\succ_i^2] \in \mathcal{P}^2$ such that x_{i1} is the first-ranked alternative and x_{i2} is the second-ranked alternative, define $\lambda^2([\succ_i^2]) = p(x_{i2}, X \setminus \{x_{i1}\}) - p(x_{i2}, X)$. It directly follows that λ^2 is a signed weight function over $\mathcal{P}^2$ that represents p_2. Next, by our inductive hypothesis, we assume that for each $k \in \{1, \ldots, n-1\}$, there is a signed weight function λ^k over $\mathcal{P}^k$ that represents p_k. Next, we show that we can construct λ^{k+1} over $\mathcal{P}^{k+1}$ that represents p_{k+1}.

Note that $\mathcal{P}^{k+1}$ is a refinement of $\mathcal{P}^k$, in which each equivalence class $[\succ^k] \in \mathcal{P}^k$ is divided into sub-equivalence classes $\{[\succ_1^{k+1}], \cdots [\succ_{n-k}^{k+1}]\} \subset \mathcal{P}^{k+1}$. Given λ^k, we require λ^{k+1} satisfy for each $[\succ^k] \in \mathcal{P}^k$ the following

$$\lambda^k([\succ^k]) = \sum_{j=1}^{n-k} \lambda^{k+1}([\succ_j^{k+1}]). \tag{17}$$

If λ^{k+1} satisfies (17), then since induction hypothesis implies that λ^k represents p_k, we get for each $S \in \Omega_k$ and $x \in S$, $p(x, S) = \lambda^{k+1}\big(\{[\succ_j] \in \mathcal{P}^{k+1} \ : x = max(S, \succ_j)\}\big)$.

Next, we show that λ^{k+1} can be constructed such that (17) holds, and for each

$S \in \Omega_{k+1} \backslash \Omega_k$, λ^{k+1} represents $p_{k+1}(S)$. To see this, pick any $S \in \Omega_{k+1} \backslash \Omega_k$. It follows that $|S| = n - k$. Let $S = \{x_1, , .., x_{n-k}\}$ and $X \setminus S = \{y_1, y_2, \cdots y_k\}$. Recall that each $[\succ^k] \in \mathcal{P}^k$ linearly orders a fixed set of k-many alternatives. Let $\{\succ^k\}$ denote the set of k alternatives ordered by $\succ^k$. Now, there exist $k!$-many $[\succ^k] \in \mathcal{P}^k$ such that $\{\succ^k\} = X \setminus S$. Let $\left\{[\succ_1^k], \cdots, [\succ_{k!}^k]\right\}$ be the collection of all such classes. Each preference that belongs to one of these classes is a different ordering of the same set of k alternatives.

Now, let $I = \{1, \ldots, k!\}$ and $J = \{1, \ldots, n-k\}$. For each $i \in I$ and $j \in J$, suppose that $\succ_{ij}^{k+1}$ linearly orders $X \setminus S$ as in $\succ_i^k$ and ranks x_j in the $k+1^{th}$ position. Consider the associated equivalence class $[\succ_{ij}^{k+1}]$. Next, we specify $\lambda^{k+1}([\succ_{ij}^{k+1}])$, the signed weight of $[\succ_{ij}^{k+1}]$, such that the resulting λ^{k+1} represents p_{k+1}. To see this, we proceed in two steps.

Step 1: First, we show that for each $S \in \Omega_{k+1} \backslash \Omega_k$, if the associated $\{\lambda_{ij}^{k+1}\}_{ij \in I \times J}$ satisfies the following two equalities for each $i \in I$ and $j \in J$,

$$\sum_{j \in J} \lambda_{ij}^{k+1} = \lambda^k([\succ_i^k]) \tag{RS}$$

$$\sum_{i \in I} \lambda_{ij}^{k+1} = q(x_j, S) \tag{CS}$$

then λ^{k+1} represents $p_{k+1}(S)$. For each $S \in \Omega$ and $x_j \in S$, $q(x_j, S)$ is as defined in (12) by using the given RCF p.

For each $S \in \Omega$ and $a \in S$, let $B(a, S)$ be the collection of all preferences at which a is the best alternative in S, and for each $k \in \mathbb{N}$ such that $n - k \leq |S|$, $\mathbf{B}^{k+1}(a, S)$ be the set of associated equivalence classes in $\mathcal{P}^{k+1}$, i.e. $B(a, S) = \{\succ \in \mathcal{P} : a = max(S, \succ)\}$ and $\mathbf{B}^{k+1}(a, S) = \{[\succ^{k+1}] \in \mathcal{P}^{k+1} : [\succ^{k+1}] \subset B(a, S)\}$. To prove the result we have to show that for each $x_j \in S$,

$$p(x_j, S) = \sum_{\{[\succ^{k+1}] \in \mathbf{B}^{k+1}(x_j, S)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{18}$$

To see this, for each $\succ \in \mathcal{P}$ and $a \in X$, let $W(\succ, a)$ denote the set of alternatives that are worse than a at $\succ$ and a itself, i.e. $W(\succ, a) = \{x \in X : a \succ x\} \cup \{a\}$. For each

$S \in \Omega$ with $a \in X$. Let $Q(a, S)$ be the collection of all preferences such that $W(\succ, a)$ is exactly $S \cup \{a\}$ and for each $k \in \mathbb{N}$ such that $n - k \leq |S|$, $\mathbf{Q}^{k+1}(a, S)$ be the set of associated equivalence classes in $\mathcal{P}^{k+1}$, i.e. $Q(a, S) = \{\succ \in \mathcal{P} : W(\succ, a) = S \cup \{a\}\}$ and $\mathbf{Q}^{k+1}(a, S) = \{[\succ^{k+1}] \in \mathcal{P}^{k+1} : [\succ^{k+1}] \subset Q(a, S)\}$. Note that, for each $x_j \in S$, we have $Q(x_j, S) = \bigcup_{i \in I} [\succ_{ij}^{k+1}]$. Moreover, it directly follows from the definitions of $Q(x_j, \cdot)$ and $B(x_j, \cdot)$ that

$$B(x_j, S) = \bigcup_{S \subset T} Q(x_j, T). \tag{19}$$

It follows from this observation that the right-hand side of (18) can be written as

$$\sum_{S \subset T} \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, T)\}} \lambda^{k+1}([\succ^{t+1}]). \tag{20}$$

i. Since (CS) holds, we have

$$q(x_j, S) = \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, S)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{21}$$

ii. Next, we argue that for each $T \in \Omega$ such that $S \subsetneq T$,

$$q(x_j, T) = \sum_{\{[\succ^{k+1}] \in \mathbf{Q}^{k+1}(x_j, T)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{22}$$

To see this, recall that by definition of $q(x_j, T)$ (12), we have

$$q(x_j, T) = \sum_{T \subset T'} (-1)^{|T'| - |T|} p(x_j, T'). \tag{23}$$

Since by the induction hypothesis, λ^k represents p_k, we have

$$p(x_j, T') = \sum_{\{[\succ^k] \in \mathbf{B}^k(x_j, T')\}} \lambda^k([\succ^k]). \tag{24}$$

Next, suppose that we substitute (24) into (23). Now, consider the set collection $\{B(x_j, T')\}_{\{T \subset T'\}}$. Note that if we apply the *principle of inclusion-exclusion* to this set collection, then we obtain $Q(x_j, T)$. It follows that

$$\sum_{T \subset T'} (-1)^{|T'| - |T|} \sum_{\{[\succ^k] \in \mathbf{B}^k(x_j, T')\}} \lambda^k([\succ^k]) = \sum_{\{[\succ^k] \in \mathbf{Q}^k(x_j, T)\}} \lambda^k([\succ^k]). \tag{25}$$

Since (RS) holds, we have

$$\sum_{\{[\succ^k]\in\mathbf{Q}^k(x_j,T)\}} \lambda^k([\succ^k]) = \sum_{\{[\succ^{k+1}]\in\mathbf{Q}^{k+1}(x_j,T)\}} \lambda^{k+1}([\succ^{k+1}]). \tag{26}$$

Thus, if we combine (23)-(26), then we obtain that (22) holds.

Now, (20) combined with (21) and (22) imply that the right-hand side of (18) equals to $\sum_{S\subset T} q(x_j,T)$. Finally, it follows from Lemma 6 that

$$p(x_j,S) = \sum_{S\subset T} q(x_j,T). \tag{27}$$

Thus, we obtain that (18) holds.

In what follows we show that for each $S \in \Omega_{k+1}\backslash\Omega_k$, there exists $k! \times (n-k)$ matrix $\lambda = [\lambda_{ij}^{k+1}]$ such that both (RS) and (CS) holds, and each $\lambda_{ij}^{k+1} \in [-1,1]$. To prove this we use Lemma 5. For this, for each $i \in I$ let $r_i = \lambda^k([\succ_i^k])$ and for each $j \in J$ let $c_j = q(x_j,S)$. Then, let $R = [r_1,\ldots,r_{k!}]$ and $C = [c_1,\ldots,c_{n-k}]$. In Step 2, we show that the sum of C equals the sum of R. In Step 3, we show that for each $k > 1$, $2k! \geq \sum_{i=1}^{k!} |r_i| + \sum_{j=1}^{n-k} |c_j|$.

Step 2: We show that the sum of C equals the sum of R, i.e.

$$\sum_{j\in J} q(x_j,S) = \sum_{i\in I} \lambda^k[\succ_i^k]. \tag{28}$$

First, if we substitute (12) for each $q(x_j,S)$, then we get

$$\sum_{j\in J} q(x_j,S) = 1 + \sum_{j\in J}\sum_{S\subsetneq T}(-1)^{|T|-|S|}p(x_j,T). \tag{29}$$

Now, let $F(x_j)$ be the collection of preferences $\succ$ such that there exists $T \in \Omega$ such that $S \subsetneq T$ and x_j is the $\succ$-best alternative in T, i.e. $F(x_j) = \{\succ \in \mathcal{P} \ : \ max(T,\succ) = x_j$ for some $S \subsetneq T\}$. For each $k \in \mathbb{N}$ such that $n-k \leq |S|$, let $\mathbf{F}(x_j)$ be the set of associated equivalence classes in $\mathcal{P}^k$. Next, we show that for each $x_j \in S$,

$$\sum_{S\subsetneq T}(-1)^{|T|-|S|+1}p(x_j,T) = \sum_{\{[\succ^k]\in\mathbf{F}(x_j)\}} \lambda^k([\succ^k]). \tag{30}$$

To see this, first, since by the induction hypothesis, λ^k represents p_k, we can replace each $p(x_j,T)$ with $\sum_{\{[\succ^k]\in\mathbf{B}^k(x_j,T)\}} \lambda^k([\succ^k])$. Next, consider the set collection

$\{B(x_j,T)\}_{\{S\subsetneq T\}}$. Since $\cup_{\{S\subsetneq T\}}B(x_j,T) = F(x_j)$, it follows from the *principle of inclusion-exclusion* that (30) holds. Next, when we substitute (30) in (29), we obtain

$$\sum_{j\in J} q(x_j,S) = 1 - \sum_{\{[\succ^k]\in\mathbf{F}(x_j)\}} \lambda^k([\succ^k]). \tag{31}$$

Then, since, by the induction hypothesis, λ^k represents p_k, we can replace 1 with $\sum_{\{[\succ^k]\in\mathcal{P}^k\}}\lambda^k([\succ^k])$. Finally, note that an equivalence class $[\succ^k]\notin\cup_{j\in J}\mathbf{F}(x_j)$ if and only if $\{\succ^k\}\cap S=\emptyset$. This means $\mathcal{P}^k\setminus\cup_{j\in J}\mathbf{F}(x_j) = \{[\succ_i^k]\}_{\{i\in I\}}$. It follows that (28) holds.

Step 3: To show that the base of induction holds, we showed that for $k=1$ and $k=2$, the desired signed weight functions exist. To get the desired signed weight functions for each $k+1>2$, we will apply Lemma 5. To apply Lemma 5, we have to show that for each $k\geq 2$, $\sum_{i=1}^{k!}|r_i| + \sum_{j=1}^{n-k}|c_j| \leq 2k!$. In what follows we show that this is true. That is, we show that for each $S\in\Omega_{k+1}\setminus\Omega_k$

$$\sum_{i\in I}|\lambda^k([\succ_i^k])| + \sum_{j\in J}|q(x_j,S)| \leq 2k!. \tag{32}$$

To see this, first we will bound the term $\sum_{i\in I}|\lambda^k([\succ_i^k])|$. As noted before, each $i\in I=\{1,\ldots,k!\}$ corresponds to a specific linear ordering of $X\setminus S$. For each $y\notin S$, there are $k-1!$ such different orderings that rank y at the kth position. So, there are $k-1!$ different equivalence classes in $\mathcal{P}^k$ that rank y at the kth position. Let $I(y)$ be the index set of these equivalence classes. Since $\{I(y)\}_{y\notin S}$ partitions I, we have

$$\sum_{i\in I}|\lambda^k([\succ_i^k])| = \sum_{y\notin S}\sum_{i\in I(y)}|\lambda^k([\succ_i^k])|. \tag{33}$$

Now, fix $y\notin S$ and let $T=S\cup\{y\}$. Since for each $i\in I(y)$, $[\succ_i^k]\in\mathbf{Q}^k(y,T)$ and vice versa, we have

$$\sum_{i\in I(y)}|\lambda^k([\succ_i^k])| = \sum_{[\succ_i^k]\in\mathbf{Q}^k(y,T)}|\lambda^k([\succ_i^k])|. \tag{34}$$

Recall that by the definition of $q(y,T)$, we have

$$q(y,T) = \sum_{[\succ_i^k]\in\mathbf{Q}^k(y,T)}\lambda^k([\succ_i^k]). \tag{35}$$

Next, consider the construction of the values $\{\lambda^k([\succ_i^k]\}_{\{i\in I(y)\}}$ from the previous step. For $k=2$, as indicated in showing the base of induction, there is only one row; that is, there is a single $\{[\succ_i^k]\} = \mathbf{Q}^k(y,T)$. Therefore, we directly have $|\lambda^k([\succ_i^k])| = |q(y,T)|$. For $k>2$, we construct λ^k by applying Lemma 5. It follows from iii of Lemma 5 that

$$\sum_{[\succ_i^k]\in\mathbf{Q}^k(y,T)} |\lambda^k([\succ_i^k])| \leq |q(y,T)| + \frac{(k-1)!}{n-k+1}. \tag{36}$$

Now, if we sum (36) over $y \notin S$, we get

$$\sum_{y\notin S} \sum_{[\succ_i^k]\in\mathbf{Q}^k(y,S\cup y)} |\lambda^k([\succ^k])| \leq \left(\sum_{y\notin S} |q(y,S\cup y)|\right) + \frac{k!}{n-k+1}. \tag{37}$$

Recall that by definition, we have $\mathbf{Q}^k(y,S\cup y) = \mathbf{Q}^k(y,S)$ and $q(y,S\cup y) = q(y,S)$. Similarly, since each $j \in J = \{1,\ldots,n\}$ denotes an alternative $x_j \in S$, we have $\sum_{x\in S}|q(x,S)| = \sum_{j\in J}|q(x_j,S)|$. Now, if we add $\sum_{j\in J}|q(x_j,S)|$ to both sides of (37), then we get

$$\sum_{i\in I} |\lambda^k([\succ_i^k])| + \sum_{j\in J} |q(x_j,S)| \leq \sum_{x\in X} |q(x,S)| + \frac{k!}{n-k+1}. \tag{38}$$

Since by Lemma 7, $\sum_{x\in X}|q(x,S)| \leq 2^k$, we get

$$\sum_{i\in I} |\lambda^k([\succ_i^k])| + \sum_{j\in S} |q(x_j,S)| \leq 2^k + \frac{k!}{n-k+1}. \tag{39}$$

Finally, note that since for each k such that $2<k<n$ $2^k \leq \frac{(2n-2k+1)k!}{n-k+1}$ holds, we have $2^k + \frac{k!}{n-k+1} \leq 2k!$. This, together with (39), implies that (32) holds. Thus, we complete the inductive construction of the desired signed weight function λ. This completes the proof.

6 Proof of Proposition 2

We leave it to the reader to show that if an RCF p is prudential w.r.t. a given $(\succ_1,\succ_2)$, then p satisfies our axioms. Conversely, let p be an RCF that satisfies our axioms. Before constructing the weight function, let us make a key observation. Consider the

Type 0		Type 1		Type 2		Type 3		Type 4	
$\succ_1$	$\succ_2$	$\succ_1$	$\succ_2$	$\succ_1$	$\succ_2$	$\succ_1$	$\succ_2$	$\succ_1$	$\succ_2$
x	x	x	y	x	y	x	y	x	y
y	y	y	x	y	x	w	x	y	z
				z	z	y	w	z	x

five types of *configurations* below that are obtained by restricting a given $(\succ_1, \succ_2)$ to a given choice set. To clarify the terminology, we say that *type i configuration is observed* if there is a choice set such that if we restrict the given $(\succ_1, \succ_2)$ to this set, then we obtain a configuration as in type i. For example, type 2 configuration is observed if there exist $x, y, z \in X$ such that $x >> z$ and $y >> z$, but neither $x >> y$ nor $y >> x$. For each choice set $S \in X$, if we obtain the configuration type i when $(\succ_1, \succ_2)$ are restricted to S, then S is called **a type i choice set**.

First, it is easy to note that *domination* implies that for each type i choice set S_i, if $x = max(S_i, \succ_1)$ and $y = max(S_i, \succ_2)$, then $p(x, S_i) + p(y, S_i) = 1$. Next, note that for each $S \in X$, there exists a type i choice set S_i, for $i \in \{0, \ldots, 4\}$, such that S is isomorphic to S_i. Then, it follows from *best-worst neutrality* that if we construct the weights as to obtain $p((max(S_i), \succ))$, then by using the same weights we obtain $p((max(S), \succ))$. This together with the first observation imply that to render a prudential representation for p with respect to $(\succ_1, \succ_2)$, it is sufficient to construct the weights as to generate the choice probabilities for these five types of choice sets.

Now, we need to construct four weights, namely λ_1, λ_2, λ_3, and λ_4, as to render a prudential representation of p w.r.t. $(\succ_1, \succ_2)$. Note that depending on X and $(\succ_1, \succ_2)$, we may not observe each configuration type. In what follows, we analyze the problem case by case. First let us make some primitive observations to rule out the trivial cases. If $X = \{x, y\}$, then the construction is trivial, so we assume that X has at least three alternatives. We assume that there exist distinct $x, y \in X$ with $x \succ_1 y$, and $y \succ_2 x$. If not, then $\succ_1 = \succ_2$, and *domination* implies that for each $S \in \Omega$,

$p(max(S, \succ_1)) = 1$. So, we can choose the weights in any arbitrary way. For each $S \in \Omega$ that is isomorphic to a type 0 choice set, the alternative that is $\succ_1$- and $\succ_2$-best is chosen with probability one, irrespective of the weight function. Therefore, we disregard these choice sets in the following reasoning.

Case 1: Suppose there exist $x, y, z, w \in X$ such that z is a decoy for y when x is available and w is a decoy for x when y is available. It follows that x, y, z, w are all distinct. Now, first define $\lambda_1 = p(x, \{x, y, z, w\})$ and $\lambda_2 = p(y, \{x, y, z, w\})$. Since $x >> w$ and $y >> z$, it follows from *domination* that $\lambda_1 + \lambda_2 = 1$. Next, consider the set $\{x, y, z\}$, and define $\lambda_4 = \frac{p(y,\{x,y,z\})-\lambda_2}{p(y,\{x,y,z\})}$. Since $x >> w$, *attraction* implies that $p(x, \{x, y, z, w\} \geq p(x, \{x, y, z\}$. This, together with our choice of λ_2, implies that $p(y, \{x, y, z\}) - \lambda_2 \geq 0$. Therefore, $\lambda_4 \geq 0$, and we obtain that $p(y, \{x, y, z\}) = \frac{\lambda_2}{1-\lambda_4}$, as desired. To define λ_3, consider the set $\{x, y, w\}$ and define $\lambda_3 = \frac{p(x,\{x,y,w\})-\lambda_1}{p(x,\{x,y,w\})}$. Similarly, *attraction* implies that $\lambda_3 \geq 0$, and we obtain that $p(x, \{x, y, w)\} = \frac{\lambda_1}{1-\lambda_3}$, as desired. Finally, consider the set $\{x, y\}$. It follows from *attraction gain equivalence* that if we substitute the defined weights for the choice likelihoods except $L(x, \{x, y\})$, then we obtain that $\frac{p(x,\{x,y\})}{p(y,\{x,y\})} = \frac{\lambda_1-\lambda_4}{\lambda_2-\lambda_3}$, as desired.

Case 2: Suppose for each distinct $x, y \in X$ there is no $z, w \in X$ such that $x >> w$ and $y >> z$. It follows that for each distinct $x, y \in X$ and $z, w \in X$, either z is a decoy for y when x is available or w is a decoy for x when y is available. Assume w.l.o.g. that z is a decoy for y when x is available. Now, first define $\lambda_1 = p(x, \{x, y, z\})$ and $\lambda_2 = p(y, \{x, y, z\})$. If there exists an alternative w that is a decoy for x when y is available, then define λ_3 as to satisfy $p(x, \{x, y, w\}) = \frac{\lambda_1}{1-\lambda_3}$. For a given λ_1 and λ_2, there exists a unique such λ_3. Finally, define λ_4 as to satisfy $p(x, \{x, y\}) = \frac{\lambda_1-\lambda_4}{\lambda_2-\lambda_3}$.

Case 3: Suppose that both case 1 and case 2 fail to hold. Since case 2 fails to hold, there exist distinct $x, y \in X$ and $z, w \in X$ such that $x >> w$ and $y >> z$. Since case 1 fails to hold, three scenarios can happen: (1) Both x and y dominate z and w, (2) z is a decoy for y when x is available, and $y >> w$, or (3) w is a decoy for x when y is available, and $x >> z$.

Suppose that scenario (1) holds, we follow a construction similar to that of case

2. First, define $\lambda_1 = p(x, \{x, y, z, w\})$ and $\lambda_2 = p(y, \{x, y, z, w\})$. Then, since there is no alternative that is a decoy for another in the availability of a third one, we can freely define λ_3 and λ_4 to satisfy $p(x, \{x, y\}) = \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_3}$.

Suppose that scenario (2) holds, then we follow a construction similar to that of case 1. First, define $\lambda_1 = p(x, \{x, y, z, w\})$ and $\lambda_2 = p(y, \{x, y, z, w\})$. Define λ_4 as to satisfy $p(y, \{x, y, z\}) = \frac{\lambda_2}{1 - \lambda_4}$. Next, since w is not a decoy for x when y is available, we can define λ_3 as to satisfy $p(x, \{x, y\}) = \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_3}$. Finally, for scenario (3), a symmetric construction works. Thus, for all possible cases, we can define a weight function λ as to render a prudential representation for p with respect to $(\succ_1, \succ_2)$.

References

Allison, G. T. (1971), *Essence of decision: Explaining the Cuban missile crisis*, number 327.5 (729.1), Little, Brown and Company,. 4

Barberá, S. & Pattanaik, P. K. (1986), 'Falmagne and the rationalizability of stochastic choices in terms of random orderings', *Econometrica: Journal of the Econometric Society* pp. 707–715. 6, 16

Block, H. & Marschak, J. (1960), *Contributions to Probability and Statistics*, Stanford University Press, Stanford, CA. 6

Bossert, W. & Sprumont, Y. (2013), 'Every choice function is backwards-induction rationalizable', *Econometrica* **81**(6), 2521–2534. 8

Brams, S. J. & Fishburn, P. C. (1978), 'Approval voting', *American Political Science Review* **72**(3), 831–847. 9

Brualdi, R. A. & Ryser, H. J. (1991), *Combinatorial Matrix Theory*, Vol. 39, Cambridge University Press. 17, 31

Debreu, G. (1960), 'Review of individual choice behaviour', *American Economic Review* **50**, 186–88. 8, 13

Echenique, F., Saito, K. & Tserenjigmid, G. (2013), The perception-adjusted Luce model. mimeo. 14

Eliaz, K., Richter, M. & Rubinstein, A. (2011), 'Choosing the two finalists', *Economic Theory* **46**(2), 211–219. 23

Falmagne, J.-C. (1978), 'A representation theorem for finite random scale systems', *Journal of Mathematical Psychology* **18**(1), 52–72. 6, 7, 16, 17

Felsenthal, D. S. (1989), 'On combining approval with disapproval voting', *Systems Research and Behavioral Science* **34**(1), 53–60. 9

Fiorini, S. (2004), 'A short proof of a theorem of Falmagne', *Journal of mathematical psychology* **48**(1), 80–82. 35

Ford Jr, L. R. & Fulkerson, D. R. (2015), *Flows in networks*, Princeton University Press. 6, 15, 16, 17, 31

Franklin, B. (1887), *The Complete Works of Benjamin Franklin: 1772-1775*, Vol. 5, GP Putnam's Sons. 3

Gelb, L. H. & Betts, R. K. (2016), *The irony of Vietnam: The system worked*, Brookings Institution Press. 4

Glass, B. (1988), *The Correspondence of Charles Darwin. Volume 2: 1837-1843*, University of Chicago Press. 3

Gul, F., Natenzon, P. & Pesendorfer, W. (2014), 'Random choice as behavioral optimization', *Econometrica* **82**(5), 1873–1912. 13

Heller, I. & Tompkins, C. (1956), 'An extension of a theorem of dantzig's', *Linear inequalities and related systems* **38**, 247–252. 26

Hoffman, A. J. & Kruskal, J. B. (2010), Integral boundary points of convex polyhedra, *in* '50 Years of Integer Programming 1958-2008', Springer, pp. 49–76. 26

Huber, J. & Puto, C. (1983), 'Market boundaries and product choice: Illustrating attraction and substitution effects', *The Journal of Consumer Research* **10(1)**, 31–44. 14

Kalai, G., Rubinstein, A. & Spiegler, R. (2002), 'Rationalizing choice functions by multiple rationales', *Econometrica* **70**(6), 2481–2488. 8

Luce, R. (1959), *Individual choice behavior: a theoretical analysis*, Wiley, New York. 13

Luce, R. D. & Raiffa, H. (1957), 'Games and decisions: Introduction and critical survey (new york, 1957)', *Chs. vi and xiv* . 25

Manzini, P. & Mariotti, M. (2014), 'Stochastic choice and consideration sets', *Econometrica* **82**(3), 1153–1176. 8

Manzini, P. & Mariotti, M. (2017), 'Dual random utility maximisation'. 24

Marley, A. A. & Louviere, J. J. (2005), 'Some probabilistic models of best, worst, and best–worst choices', *Journal of Mathematical Psychology* **49**(6), 464–480. 9

McClellon, M. (2015), 'Non-additive random utility functions'. 21

McFadden, D. (1978), 'Modeling the choice of residential location', *Transportation Research Record* (673). 6

McGarvey, D. C. (1953), 'A theorem on the construction of voting paradoxes', *Econometrica: Journal of the Econometric Society* pp. 608–610. 7, 29

Natenzon, P. (n.d.), 'Random choice and learning', *Journal of Political Economy* p. forthcoming. 14

Payne, J. W. & Puto, C. (1982), 'Adding asymmetrically dominated alternatives: Violations of regularity and the similarity hypothesis', *Journal of Consumer Research* **9(1)**, 90–98. 14

Rieskamp, J., Busemeyer, J. R. & Mellers, B. A. (2006), 'Extending the bounds of rationality: evidence and theories of preferential choice', *Journal of Economic Literature* pp. 631–661. 14

Saito, K. (2017), 'Axiomatizations of the mixed logit model'. 9, 15, 17

Shafir, E. (1993), 'Choosing versus rejecting: Why some options are both better and worse than others', *Memory & cognition* **21**(4), 546–556. 5

Shafir, E., Simonson, I. & Tversky, A. (1993), 'Reason-based choice', *Cognition* **49**(1), 11–36. 4

Stanley, R. P. (1997), *Enumerative Combinatorics. Vol. 1, vol. 49 of Cambridge Studies in Advanced Mathematics*, Cambridge University Press. 35

Telhami, S. (1990), 'Israeli foreign policy: A static strategy in a changing world', *Middle East Journal* **44**(3), 399–416. 4

Tversky, A. (1972), 'Elimination by aspects: A theory of choice.', *Psychological review* **79**(4), 281. 11