On Acceptant and Substitutable Choice Rules*

Battal Doğan† Serhat Doğan ‡ Kemal Yıldız§

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Abstract

Each *acceptant* and *substitutable* choice rule is known to have a maximizer-collecting representation: there exists a list of priority orderings such that from each choice set that includes more elements than the capacity, the choice is the union of the priority orderings' maximizers (Aizerman and Malishevski, 1981). We introduce the notion of a prime atom and constructively prove that the number of prime atoms of a choice rule determines its smallest size maximizer-collecting representation. We show that responsive choice rules require the maximal number of priority orderings in their maximizer-collecting representations among all *acceptant* and *substitutable* choice rules. We characterize maximizer-collecting choice rules in which the number of priorities equals the capacity. We also show that if the capacity is greater than three and the number of elements exceeds the capacity by at least two, then no *acceptant* and *substitutable* choice rule has a maximizer-collecting representation of the size equal to the capacity.

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†Department of Economics, University of Bristol; battal.dogan@bristol.ac.uk.

‡Department of Economics, Bilkent University; dserhat@bilkent.edu.tr.

§Department of Economics, Bilkent University; kemal.yildiz@bilkent.edu.tr.

Contents

1 Introduction **3**

1.1 Related Literature 5

2 Preliminaries **6**

3 A canonical representation of acceptant and substitutable choice rules **9**

3.1 Maximizer-collecting representation of responsive choice rules 18

4 Maximizer collecting choice rules of size q **25**

5 Conclusion **30**

1 Introduction

Recent advances in market design has called for a better understanding of how institutions choose, or how they should choose, when faced with a set of elements. For example, in the context of assigning students to schools, it is important to understand the structure of plausible choice rules a school can use as an admissions policy. Although the relevant restrictions on choice rules vary across applications, *acceptant* and *substitutable* choice rules remain as the general prominent class. In this study, we provide a canonical representation of *acceptant* and *substitutable* choice rules which provides a comprehensive language that conceptually organizes these choice rules.

We consider a decision maker who has a capacity constraint and encounters choice problems each of which consists of a choice set. A choice rule, at each possible choice problem, chooses some elements from the choice set without exceeding the capacity. A choice rule is *acceptant* if it accepts all elements from a choice set that includes no more elements than the capacity. *Acceptance* is a natural restriction in many applications where institutions prefer to fill their positions whenever possible. A choice rule is *substitutable* if it chooses an element from a choice set whenever the element is chosen from a larger choice set.[1] That is, no element should be chosen because it complements another element. *Substitutability* has been a standard requirement in the market design literature following the seminal work of Kelso and Crawford (1982).[2] In the context of matching problems, Alkan (2001) presents the first model that studies *substitutable* choice rules together with *acceptance*.[3]

[1] In the revealed preference literature, *substitutability* is also referred to as *independence of irrelevant alternatives*, *Chernoff's condition* or *Sen's property* α.

[2] Beyond its normative appeal, Hatfield and Milgrom (2005) show that *substitutability* of choice rules guarantees the existence of a *stable* matching, which is a central desideratum for applications. Hatfield and Kojima (2008) show that *substitutability* of choice rules is an "almost necessary" condition for the non-emptiness of the core and the existence of *stable* allocations. Similarly, several classical results of matching literature have been generalized with *substitutable* choice rules (Roth and Sotomayor (1990), Alkan and Gale (2003), Hatfield and Milgrom (2005)).

[3] Alkan (2001) calls *acceptant* and *substitutable* choice rules *quotafilling* and analyze the structure of stable matchings in two-sided markets in which choice functions are *quotafilling*.

Given capacity q, we say that a choice rule has a maximizer-collecting (MC) representation of size m, or simply called m-maximizer-collecting, if there exists a list of m-many priority orderings[4] such that all elements are chosen from each choice set that contains at most q elements, and the choice from each choice set that contains more than q elements is obtainable by collecting the maximizers of the priority orderings. By Aizerman and Malishevski (1981), it is known that each *acceptant* and *substitutable* choice rule has an MC representation.[5] However, the size of a smallest size MC representation of a choice rule and how to construct such a representation have been unknown.

We introduce the concept of a prime atom of a choice rule, which turns out to determine the minimal number of priority orderings required for an MC representation of an *acceptant* and *substitutable* choice rule. Given a choice rule, a choice set is a *prime atom* if the number of elements in the choice set is equal to the capacity and there exists an element that is chosen whenever added to the choice set, but no longer chosen whenever any other element is added afterwards (see Section 1 for a formal definition). In Theorem 1, we constructively prove that for each *acceptant* and *substitutable* choice rule, the number of priority orderings in its smallest size MC representation is equal to the number of its prime atoms.

Well-known examples of *acceptant* and *substitutable* choice rules include responsive choice rules which have been studied particularly in the two-sided matching context (Gale and Shapley, 1962). A choice rule is responsive if there exists a priority ordering such that the choice from each choice set is obtainable by choosing the highest priority elements until the capacity is reached or no element is left. In Proposition 2, we show that the upper bound on the number of prime atoms is achieved by responsive choice rules. That is, responsive choice rules render an MC representation of the largest size among all *acceptant* and *substitutable* choice rules.

[4] A priority ordering is a complete, transitive, and anti-symmetric binary relation over all possible elements.

[5] Aizerman and Malishevski (1981) provide their representation result for *path independent* choice rules (Plott, 1973). *Acceptance* and *substitutability* imply *path independence*.

In the second part of the paper, we analyze q-MC choice rules. For applications, q-MC choice rules have a particular appeal. If a choice rule is q-MC, then each one of the q priority orderings in its MC representation can be interpreted as a separate admission criterion for each available position. Put differently, the admission criterion for each position is represented by the associated priority ordering in the MC representation of size q. However, in Theorem 3, we show that if the capacity is greater than three and the number of elements exceeds the capacity by at least two, then no *acceptant* and *substitutable* choice rule has an MC representation of size q. This impossibility result has an implication in the context of school choice. To achieve diversity, schools typically come up with slot specific priorities and choose students from a choice set in a lexicographic fashion (Kominers and Sönmez, 2016). If a school with q-many slots could have a q-MC choice rule, then it could transparently reveal the criterion to be used at each slot. Moreover, since a q-MC choice rule makes the same choice independent of the precedence order to be followed in filling the slots, it is immune to, rather debatable, precedence effects.[6] However, Theorem 3 provides a sense in which such precedence effects are unavoidable.

1.1 Related Literature

Acceptance together with *substitutability* imply *path independence* (Plott, 1973), which requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set. Among others, Plott (1973), Moulin (1985), Johnson (1990), and Johnson (1995) study the structure of *path independent* choice rules. Johnson and Dean (2001) and Koshevoy (1999) provide a lattice theoretic characterization of *path independent* choice rules.[7]

[6] In this vein, Dur et al. (2013) and Dur et al. (2018) provide empirical evidence, by using school choice data from Boston school district, indicating that the order in the priority profile may cause additional, possibly unintended, advantages for some groups of students.

[7] They show that choice lattices associated with these rules constitute the class of lower locally distributive lattices.

Chambers and Yenmez (2017) use the MC representation of path independent choice rules to provide a new proof of a classical existence result for stable matching and a new result on welfare effects of expanding the choice rules in the matching context. Kojima and Manea (2010), Ehlers and Klaus (2014), and Ehlers and Klaus (2016) characterize deferred acceptance mechanisms where each school has a choice rule that satisfies *acceptance* and *substitutability*. Although the structure of *acceptant* and *substitutable* choice rules and their relation to matching mechanisms have been extensively studied, there is no direct implication of these studies for our construction of priority orderings that render a canonical MC representation.

As for our analysis of q-MC choice rules, the closest study is by Eliaz et al. (2011), who axiomatically analyze a decision maker who has in mind two orderings and chooses one or two elements that are maximizers of these orderings. This procedure, called top-and-the-top, yields a distinct pair of elements from each choice set, only if the second ordering is the inverse of the first ordering. In contrast, for any given capacity q, a q-MC choice rule chooses a distinct set of q-many elements from each choice set that contains at least q elements.

2 Preliminaries

Let A be a nonempty finite set of n elements and let $\mathcal{A}$ denote the set of all nonempty subsets of A. A **choice rule** $C : \mathcal{A} \to \mathcal{A}$ associates with each choice set $S \in \mathcal{A}$, a nonempty set of elements $C(S) \subseteq S$. Let $q \in \{1, \ldots, n\}$ be a given capacity. We analyze choice rules that satisfy the following two properties that are well-known in the literature.

(q-)Acceptance: For given capacity $q \in \mathbb{N}$, an element is rejected from a choice set at a capacity q only if the capacity is full. Formally, for each $S \in \mathcal{A}$,

$$|C(S)| = \min\{|S|, q\}.$$

Substitutability: If an element is chosen from a choice set, then it is also chosen from any subset of the choice set that contains the element. Formally, for each $S, T \in \mathcal{A}$

such that $T \subseteq S$ and $a \in T$,

$$\text{if } a \in C(S), \text{ then } a \in C(T).$$

Each q-*acceptant* choice rule C satisfies *substitutability* if and only if C satisfies *path independence*[8] which requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set (Plott, 1973). Formally, for each $S, S' \in \mathcal{A}$, $C(S \cup S') = C(C(S) \cup C(S'))$.

The fact that q-*acceptance* together with *substitutability* imply *path-independence* follows from two facts. The first fact is that a choice rule is *path-independent* if and only if it is *substitutable* and satisfies *irrelevance of rejected elements* (*IRE*) (Aizerman and Malishevski, 1981). *IRE* requires that for all $S, S' \in \mathcal{A}$, $C(S) \subseteq S' \subseteq S$ implies $C(S) = C(S')$. The second fact is that q-acceptance together with *substitutability* imply *IRE*. Given $S, S' \in \mathcal{A}$ such that $C(S) \subseteq S' \subseteq S$, *substitutability* implies that $C(S) \subseteq C(S')$ and q-*acceptance* implies that $|C(S)| \geq |C(S')|$, which together imply that $C(S) = C(S')$.

Aizerman and Malishevski (1981) show that a choice rule is *path independent* if and only if there exists a list of priority orderings such that the choice from each choice set is the union of the highest priority elements in the priority orderings.[9] Next, we formally define and add more structure on these choice rules that we call MC choice rules.

A **priority ordering** $\succ$ is a complete, transitive, and anti-symmetric binary relation over A. A **priority profile** $\pi = (\succ_1, \ldots, \succ_m)$, for some $m \in \mathbb{N}$, is an ordered list of m **distinct** priority orderings. Let Π denote the set of all priority profiles. Given $S \in \mathcal{A}$ and a priority ordering $\succ$, let $\max(S, \succ) = \{a \in S : \forall b \in S \setminus \{a\}, a \succ b\}$.

A choice rule C has a **maximizer-collecting (MC) representation** of size $m \in \mathbb{N}$

[8]This is also noted in Remark 1 of Doğan and Klaus (2018), and it follows from Lemma 1 of Ehlers and Klaus (2016) together with Corollary 2 of Aizerman and Malishevski (1981).

[9]In the words of Aizerman and Malishevski (1981), each *path independent* choice rule is generable by some mechanism of collected extremal choice.

(or simply m-**maximizer-collecting**) if there exists $(\succ_1, \ldots, \succ_m) \in \Pi$ such that for each $S \in \mathcal{A}$ with $|S| \leq q$, $C(S) = S$ and for each $S \in \mathcal{A}$ with $|S| > q$, $C(S)$ is obtained by collecting the maximizers of the priority orderings in S, that is,

$$C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i).$$ [10]

Next, we give examples of well-known *acceptant* and *substitutable* choice rules used in school choice applications.

Example 1 A choice rule C is **responsive** if there exists a priority ordering $\succ$ such that for each $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ$-priority elements until the capacity q is reached or no element is left.[11] Responsive choice rules have been studied particularly in the two-sided matching context (Gale and Shapley, 1962). The school choice literature, starting with the seminal study by Abdulkadiroğlu and Sönmez (2003), has widely focused on problems where each school is endowed with a responsive choice rule.

Example 2 A choice rule C is **lexicographic** if there exists a list of priority orderings $(\succ_1, \ldots, \succ_q) \in \Pi$ such that for each choice set $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ_1$-priority element in S, then choosing the highest $\succ_2$-priority element among the remaining elements, and so on until q elements are chosen or no element is left. As argued in detail by Kominers and Sönmez (2016), lexicographic choice rules have been particularly useful in designing allocation mechanisms for school choice to achieve diversity.[12]

[10] One can also consider maximizer-collecting* representation such that for each $S \in \mathcal{A}$ with $|S| \leq q$, we also require $C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i)$. As argued in Remark 1, we can generalize our Theorem 1 for maximizer-collecting* representation. However, we prefer the current presentation, since under *acceptance* asking for a maximizer-collecting* representation does not change the rules that are m-MC, but requires adding, in a sense "unnecessary", priorities to the representation.

[11] Chambers and Yenmez (2018) show that a choice rule satisfies *acceptance* and the *weakened weak axiom of revealed priority (WWARP)* if and only if it is responsive.

[12] See Doğan et al. (2017) for an axiomatic characterization of lexicographic choice rules in the school choice context.

Example 3 Echenique and Yenmez (2015) consider a school choice problem in which students are partitioned into different types, each type t has its reserved seats, and there is a common priority ordering $\succ$, such as exam scores, that ranks all the students. A choice rule is **generated by reserves for priority** $\succ$ if for each type t, the highest priority students among type-t students are chosen until the reserves for type t are filled, or type-t students are exhausted. Then for the remaining seats, $\succ$-best students are chosen until all the seats or students are exhausted, that is students of all types compete against each other for all the seats that are not filled in the first stage. It follows from the characterization of Echenique and Yenmez (2015) that these choice rules satisfy *acceptance* and *substitutability*.

3 A canonical representation of acceptant and substitutable choice rules

Although it follows from Aizerman and Malishevski (1981) that an *acceptant* choice rule C satisfies *substitutability* if and only if C is MC, they remain silent about the minimal size of the MC representation and construction of the priority profile. In this section, we construct a canonical representation of *acceptant* and *substitutable* choice rules.

We introduce the concept of a prime atom[13] of a choice rule, which will be the key in finding the minimal number of priorities needed for an MC representation. Given a choice rule C, a choice set is a prime atom of C if the number of elements in the choice set is equal to the capacity and there exists an element not belonging to the choice set that is chosen whenever added to the choice set, but no longer chosen whenever any other element is added afterwards. The formal definition is as follows.

Definition 1 *A choice set $T \in \mathcal{A}$ is a **prime atom** if $|T| = q$, and there exists an element $a \notin T$ such that $a \in C(T \cup \{a\})$ and for each $b \notin T \cup \{a\}$, $a \notin C(T \cup \{a, b\})$.*

[13]The "atom" terminology is borrowed from order theory that we utilize in the proof of Theorem 1. See Footnote 17 for clarification.

The first result shows that given an *acceptant* and *substitutable* choice rule, the number of its prime atoms determines the smallest size MC representation of the choice rule.

Theorem 1 *For each acceptant and substitutable choice rule C,*

i. C has an MC representation of a size equal to the number of its prime atoms.

ii. C does not have an MC representation of any size smaller than the number of its prime atoms.

First, we provide a sketch of the proof. The proof uses a specific choice lattice induced by a given *path independent* choice rule. To the best of our knowledge it was first noted by Johnson (1990) that each *path independent* choice rule induces this specific choice lattice.[14] We present the choice lattice in a different formal frame, which provides a useful language in proving our result. Given an *acceptant* and *substitutable* choice rule C, the associated choice lattice is $(\mathcal{M}, \searrow)$, where $\mathcal{M}$ stands for, what we call, the set of *maximal choice sets* and $\searrow$ stands for a binary relation that we call the ancestor relation. A choice set is maximal if there is no larger choice set in which the same set of elements is chosen. A maximal choice set S is a parent of another choice set S' if S' can be obtained from S by removing one of the chosen elements. A maximal choice set S is an ancestor of another choice set S' if there is a path that connects S to S' through parental relations. The collection of maximal choice sets endowed with *the ancestor relation* forms our choice lattice associated with C. For a pictorial representation of $(\mathcal{M}, \searrow)$ see the example in Section 4. In Lemmas 1-3, we prove some elementary but useful properties of $(\mathcal{M}, \searrow)$.

We observe that we can associate a priority ordering with each chain in $(\mathcal{M}, \searrow)$ that connects A to a maximal choice set of cardinality q. At this point, it is clear that collection of all such possible priority orderings provides an MC representation of C,

[14] We are grateful to Ahmet Alkan for bringing this construct to our awareness. Chambers and Yenmez (2017) use a similarly defined choice lattice in their proofs.

which in turn provides a proof of the (Aizerman and Malishevski, 1981) representation result. However, this obtained representation is not necessarily a minimal size representation.

The essential part of the proof identifies which orderings are critical for an MC representation and how to use them to construct an MC representation. We discover that some maximal choice sets that we call primes are critical for representing the choice rule. That is, if we pick a collection of chains in $(\mathcal{M}, \searrow)$ that spans the primes, i.e., for each prime there is a chain that contains the prime, then the associated priority profile represents the choice rule (see Lemma 7).

Now, it remains to relate these chains to the *prime atoms*. It follows from Lemma 4 that a choice set is a prime atom if and only if it is a prime choice set with q elements. Then, in Lemma 5 we show that each prime has a unique prime child. It follows that if we collect each chain connecting A to a prime atom, then this set of chains spans all the primes.[15] Then, by using Lemma 7, we show that the associated priority profile renders an MC representation of the given choice rule.[16] Since each ordering is associated with a single prime atom, the number of orderings equals the number of prime atoms. Finally, we also show that the number of priority orderings to represent C is at least the number of prime atoms. Next, we present the proof of Theorem 1.

Proof of Theorem 1. Part i. Let C be an *acceptant* and substitutable choice rule. We first define some notions, then introduce some lemmas, and finally present the representation result. Note that C satisfies path independence as well (See Section

[15]This conclusion does not hold with arbitrary path independent choice rules. We present an example in Footnote 18.

[16]There are two degree of freedoms in choosing these representations. One is that the last q alternatives can be ranked in any way. The other is the following. Suppose there is a prime node S which does not have a prime parent (if there is no prime, then the representation is unique up to the ranking of the last q alternatives). Now, there is exactly one priority in the representation that corresponds to the chain containing S. The part of this chain which connects S to a prime atom is unique (by Lemma 6). However the remaining part of the chain that connects S to the grand set A can be chosen in any arbitrary way. This is the second degree of freedom in the construction.

2).

A choice set is called "maximal" if it includes any other choice set from which the same set of elements is chosen. Formally, a choice set $S \in \mathcal{A}$ is **maximal** for C if for each choice set $S' \in \mathcal{A} \setminus S$ such that $C(S') = C(S)$, we have $S' \subset S$. Let $\mathcal{M}$ denote the set of maximal choice sets for C.

We define the following binary relations on $\mathcal{M}$. For each $S, S' \in \mathcal{M}$, S is a **parent** of S', denoted by $S \to S'$, if there exists $a \in C(S)$ such that $S' = S \setminus \{a\}$. For each $S, S' \in \mathcal{M}$, S is an **ancestor** of S', denoted by $S \searrow S'$, if there exists a collection of sets in $S_1, \ldots, S_k \in \mathcal{M}$ such that $S \to S_1 \to \cdots \to S_k \to S'$. Since the binary relation $\searrow$ is transitive, $(\mathcal{M}, \searrow)$ is a partially ordered set.

Each subset of $\mathcal{M}$ that is linearly ordered according to $\searrow$ is called a **chain** in $(\mathcal{M}, \searrow)$. By adopting the terminology from order theory, we call each $T \in \mathcal{M}$ such that $|T| = q$ as an **atom** of C.[17] A choice set $S \in \mathcal{M}$ is a **prime** of C if S has a unique parent, that is, there exists a unique $S' \in \mathcal{M}$ such that $S' \to S$. Let $\mathcal{P}$ denote the set of all primes of C. A collection of primes $S_1, \ldots, S_k \in \mathcal{P}$ such that $S_1 \to \cdots \to S_k$ is called a **prime chain** from S_1 to S_k. An atom of C that is also a prime of C is called a **prime atom** of C.

Lemma 1 *For each choice set $S \in \mathcal{A}$ such that $|S| = q$, there exists a unique set $S' \in \mathcal{M}$ such that $C(S') = S$.*

Proof. For each $S_1, S_2 \in \mathcal{A}$, if $C(S_1) = C(S_2) = S$, then by path independence, $C(S_1 \cup S_2) = C(C(S_1) \cup C(S_2))$, which implies $C(S_1 \cup S_2) = S$. It follows that $S' = \bigcup\{S_0 : C(S_0) = S\}$ and it is unique. ■

Lemma 2 *For each choice set $S \in \mathcal{M}$ such that $|S| > q$, and each $a \in C(S)$, we have $S \setminus \{a\} \in \mathcal{M}$.*

[17] An **atom** in a partially ordered set is an element that is minimal among all elements that are unequal to the least element. Since for constructing the desired priority profile, the choice sets with at most q alternatives do not play any role, we consider these choice sets as the least elements of $(\mathcal{M}, \searrow)$. Then we call each choice set $T \in \mathcal{M}$ with $q + 1$ alternatives an atom of $(\mathcal{M}, \searrow)$.

Proof. By contradiction suppose there exists $S'' \in \mathcal{M}$ with $S \setminus \{a\} \subsetneq S''$ and $C(S'') = C(S \setminus \{a\})$. Now, consider the set $S'' \cup \{a\}$, by path independence, we have $C(S'' \cup \{a\}) = C(C(S'') \cup \{a\}) = C(C(S)) = C(S)$. Since $S \subsetneq S'' \cup \{a\}$, this contradicts that $S \in \mathcal{M}$, It follows that $S \setminus \{a\} \in \mathcal{M}$. ■

Lemma 3 *If a maximal choice set nests another, then there is a path that connects the two. That is, for each $S, S' \in \mathcal{M}$ such that $S' \subset S$, we have $S \searrow S'$.*

Proof. Let $S, S' \in \mathcal{M}$ such that $S' \subset S$. Since $S, S' \in \mathcal{M}$ and C satisfies path independence, there exists $x_0 \in C(S) \setminus S'$. By contradiction, suppose $C(S) \subset S'$.[Then $C(S) \subset S' \subset S$, then by path independence $C(S') = C(S)$ should hold as well, so $S' \notin \mathcal{M}$, this contradicts Lemma 1] It follows from path independence that $C(C(S) \cup S') = C(C(S) \cup C(S')) = C(S \cup S') = C(S)$. Since $C(S) \cup S' = S'$, we get $C(S) = C(S')$. Since $S, S' \in \mathcal{M}$, this contradicts Lemma 1, therefore $C(S) \not\subset S'$. Now, let $S_1 = S \setminus \{x_0\}$. It follows from Lemma 2 that $S_1 \in \mathcal{M}$, and $S' \subset S_1$ implies there exists $x_1 \in C(S_1) \setminus S'$. By proceeding similarly we obtain a path $\{S_1, \ldots, S_k\}$ that connects S to S' which means $S \searrow S'$. ■

Lemma 4 *Suppose that $S \in \mathcal{M}$ and $S \cup \{a\}$ is a parent of S. Then, S is a prime if and only if a is no longer chosen whenever any other element is added to $S \cup \{a\}$, that is for each $b \notin S \cup \{a\}$, $a \notin C(S \cup \{a, b\})$.*

Proof. (*If part*) We prove the contrapositive statement. Assume that S is not a prime. Then, there exists $b \notin S \cup \{a\}$ such that $S \cup \{b\}$ is a parent of S. Since $S \cup \{a\}$ is a parent of S, we have $S \cup \{a\} \in \mathcal{M}$. Suppose that $a \notin C(S \cup \{a, b\})$. Then $C(S \cup \{a, b\}) = C(S \cup \{b\})$, which contradicts $S \cup \{b\} \in \mathcal{M}$. Hence $a \in C(S \cup \{a, b\})$.

(*Only if part*) We prove the contrapositive statement. Assume that there exists $b \notin S \cup \{a\}$ such that $a \in C(S \cup \{a, b\})$. Let $S' \cup \{a, b\} \in \mathcal{M}$ be such that $C(S \cup \{a, b\}) = C(S' \cup \{a, b\})$, and $a, b \notin S'$. Note that by the choice of S', $S \subset S'$. Since $a \in C(S' \cup \{a, b\})$, by Lemma 2, $S' \cup \{b\} \in \mathcal{M}$. Then, it follows from Lemma 3 that $S' \cup \{b\} \searrow S$. Since $a \notin S' \cup \{b\}$, the path connecting $S' \cup \{b\}$ to S must reach S via a parent other than $S \cup \{a\}$. Therefore, S is not a prime. ■

Lemma 5 *Each prime that is not an atom is the parent of a unique prime. That is, for each $S \in \mathcal{P}$ such that $|S| > q$, there exists a unique prime $S' \in \mathcal{P}$ such that $S \to S'$.*[18]

Proof. Let $S \in \mathcal{M}$ with $|S| > q$. Suppose that S is a prime. Let $S \cup \{b^*\}$ be the unique parent of S. Since $b^* \in C(S \cup \{b^*\})$ and C is *q-acceptant*, there exists unique $a \in C(S) \setminus C(S \cup \{b^*\})$. Consider the choice set $S \setminus \{a\}$. Since $S \in \mathcal{M}$ and $a \in C(S)$, it follows from Lemma 2 that $S \setminus \{a\} \in \mathcal{M}$.

We show that $S \setminus \{a\}$ is a prime. Suppose that $S \setminus \{a\}$ is not a prime. Lemma 4 implies that there exists $b \notin S$ such that $a \in C(S \cup \{b\})$. Since $a \notin C(S \cup \{b^*\})$, $b \neq b^*$. Next, consider the choice set $S \cup \{b^*, b\}$. Since S is a prime, $S \cup \{b^*\} \to S$, and $b \notin S \cup \{b^*\}$, it follows from Lemma 4 that $b^* \notin C(S \cup \{b^*, b\})$. Thus $C(S \cup \{b^*, b\}) = C(S \cup \{b\})$. Now, since $a \in C(S \cup \{b\})$, we have $a \in C(S \cup \{b^*, b\})$. But we also have $a \notin C(S \cup \{b^*\})$, contradicting that C satisfies *substitutability*. Thus, we obtain that $S \setminus \{a\}$ is a prime and $S \to S \setminus \{a\}$. That is, S has a prime child.

To see that the prime child is unique, by contradiction, suppose that there exist $a, a' \in C(S)$ such that $S \to S \setminus \{a\}$ and $S \to S \setminus \{a'\}$, where both $S \setminus \{a\}$ and $S \setminus \{a'\}$ are prime. Now, since A is not a prime, $S \neq A$, and there exists some $x \notin S$. Next, consider $S \cup \{x\}$. It follows from Lemma 4 that $a, a' \notin C(S \cup \{x\})$. This combined with C being *q-acceptant* implies there exists $y \in C(S \cup \{x\}) \setminus C(S)$. This contradicts that C satisfies *substitutability*. ■

Lemma 6 *For each prime $S \in \mathcal{P}$ that is not an atom, there exists a unique prime chain connecting S to a prime atom $T \in \mathcal{P}$. Moreover, the unique prime chain from S to T is included in any chain from A to T. Formally, for each $S \in \mathcal{P}$ such that $|S| > q$, there*

[18] As we emphasize before, this result fails if the choice rule C is *path independent* but not *acceptant*. To see this by an example, let $A = \{1, 2, 3, 4\}$, and consider the following two priorities: $1 \succ 2 \succ 3 \succ 4$ and $1 \succ' 4 \succ' 3 \succ' 2$. Let C be the maximizer collecting of $\succ$ and $\succ'$. It directly follows that C is *path independent*. But, since $C(A) = \{1\}$ and $C(A \setminus \{1\}) = \{2, 4\}$, C is not *acceptant*. To see that C fails to satisfy the conclusion of the Lemma, note that $\{2, 3, 4\}, \{2, 3\}, \{3, 4\}$ are prime choice sets. However, since both $\{2, 3, 4\} \to \{3, 4\}$ and $\{2, 3, 4\} \to \{2, 3\}$, $\{2, 3, 4\}$ is the parent of two primes. Note that A having four elements does not play any essential role in this argument, so we can construct similar examples for any universal set of alternatives.

exists a unique list $S_1, \ldots, S_k \in \mathcal{P}$ such that $S \to S_1 \cdots \to S_k \to T$, where T is a prime atom and for each $t \in \{1, \ldots, k\}$, S_t is included in each chain from A to T.

Proof. By Lemma 5, there exists a unique prime $S_1 \in \mathcal{P}$ such that $S \to S_1$. Applying Lemma 5 consecutively, there exists a unique list $S_1, \ldots, S_k \in \mathcal{P}$ such that $S \to S_1 \cdots \to S_k \to T$, where T is a prime atom. Now, since for each $t \in \{1, \ldots, k\}$, S_t is prime and T is a prime atom, each S_t and T has a unique parent. It follows that for each $t \in \{1, \ldots, k\}$, S_t must be included in any chain that connects A to T. ■

Lemma 7 *For each nonprime maximal choice set S and its parent $S \cup \{a\}$, there exists a maximal choice set $S' \in \mathcal{M}$ such that $S \cup \{a\} \subsetneq S'$ and $a \in C(S')$.*

Proof. Suppose that S is a nonprime maximal choice set and $S \cup \{a\} \to S$. Since S is not a prime, there exists $b \notin S \cup \{a\}$ such that $S \cup \{b\} \to S$. Thus, $b \in C(S \cup \{b\})$. Now, consider the choice set $S \cup \{a, b\}$. Suppose that $a \notin C(S \cup \{a, b\})$. Since C satisfies *path independence*, $C(S \cup \{a, b\}) = C(S \cup \{b\})$, which is a contradiction since $S \cup \{b\} \in \mathcal{M}$. Hence, $a \in C(S \cup \{a, b\})$. Now, by Lemma 1, there exists $S' \in \mathcal{M}$ such that $C(S') = C(S \cup \{a, b\})$. Since S' is maximal, $S \cup \{a, b\} \subset S'$. Since $a \in C(S \cup \{a, b\}) = C(S')$ and C satisfies *substitutability*, $a \in C(S')$. Thus, S' is the choice set with the desired properties. ■

Now, we are ready to construct the set of priority orderings that renders the desired representation. For each prime atom $T \in \mathcal{P}^{\mathcal{A}}$, we will associate a priority ordering with a chain that connects A to T. Let $T \in \mathcal{P}^{\mathcal{A}}$ and let $S_1, \ldots, S_k \in \mathcal{M}$ be such that $A \to S_1 \to \cdots \to S_k \to T$ is a chain that connects A to T. Let $a_1 = A \setminus S_1$, $a_{k+1} = S_k \setminus T$, and for each $i \in \{2, \ldots, k\}$, $a_i = S_{i-1} \setminus S_i$. Note that by definition of a parent, for each $i \in \{1, \ldots, k+1\}$, a_i is well-defined. Now, let $\succ^T$ be such that for each $i, j \in \{1, \ldots, k+1\}$, $a_i \succ^T a_j$ if $i < j$, and assume that any other remaining element is ranked below a_{k+1} arbitrarily. A priority profile $(\succ^T)_{T \in \mathcal{P}^{\mathcal{A}}}$ is constructed similarly.

Since each $S \in \mathcal{M}$ such that $A \to S$ is a prime, we obtain at least q-many priority orderings, i.e. $|\mathcal{P}^{\mathcal{A}}| \geq q$. Otherwise, by Lemma 6, there would be two different primes

S and S' such that $A \to S$, $A \to S'$, and there would be prime chains connecting S and S' to the same prime atom T, which would be a contradiction since it would imply that the prime atom T has at least two parents.

We show that for each $S \in \mathcal{A}$ such that $|S| > q$,

$$C(S) = \bigcup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T) \tag{1}$$

It is sufficient to show that (1) holds for each maximal set. So, let $S \in \mathcal{M}$ be such that $|S| > q$.

First, we show that $\bigcup_{T \in \mathcal{P}^{\mathcal{A}}} \max(A, \succ^T) \subset C(S)$. Let $T \in \mathcal{P}^{\mathcal{A}}$. For each $a \in A$, if $a = max(S, \succ^T)$, then by the construction of $\succ^T$, there exists $S^* \in \mathcal{M}$ such that $a \in C(S^*)$ and $S \subset S^*$. Since C satisfies *substitutability*, $a \in C(S)$.

Next, we show that $C(S) \subset \bigcup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T)$. The proof is by induction on the cardinality of S. Suppose that $|S| = n$, that is, $S = A$. Let $a \in C(S)$. Note that $S' = A \setminus \{a\}$ is a prime. If S' is a prime atom, which is the case if and only if $q = n-1$, then $a = \max(S, \succ^{S'})$. If S' is a prime that is not an atom, then by Lemma 6, there exists a unique prime chain connecting S to a prime atom $T \in \mathcal{P}$. Moreover, again by Lemma 6, the unique prime chain from S to T is included in any chain from A to T. Hence, $a = \max(S, \succ^T)$.

Now, let k be such that $n > k > q + 2$ and assume that for each $S \in \mathcal{M}$ with $|S| \geq k$, our induction hypothesis is true, that is, $C(S) \subset \bigcup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T)$. Next, we show that the hypothesis is true also for each set of cardinality $k - 1$.

Let $S \in \mathcal{M}$ be such that $|S| = k - 1$. Let S' be a parent of S, i.e. $S' \in \mathcal{M}$ with $S' \to S$. Let $a \in A$ and suppose that $S' = S \cup \{a\}$. Since $|S'| = k$, by the induction hypothesis, we know that (1) holds for S'. Since C satisfies *substitutability*, we have $C(S') \setminus \{a\} \subset C(S)$ and $C(S') \setminus \{a\} \subset \bigcup_{T \in \mathcal{P}^{\mathcal{A}}} \max(S, \succ^T)$. Now, let $x \in C(S) \setminus C(S')$. Next, we show that there exists $T \in \mathcal{P}^{\mathcal{A}}$ such that $x \in max(S, \succ^T)$. We consider two cases.

Case 1: Suppose that S is a prime. It follows from Lemma 2 that there is a unique $S'' \in \mathcal{P}$ such that $S \to S''$. Since, by *acceptance*, x is the only element in S

that is chosen in S but not chosen in S', by Lemma 4, we have $S'' = S \setminus \{x\}$. If S'' is a prime atom, then $x = \max(S, \succ^{S''})$. If S'' is a prime that is not an atom, then by Lemma 6, there exists a unique prime chain connecting S'' to a prime atom $T \in \mathcal{P}$. Moreover, again by Lemma 6, the unique prime chain from S'' to T is included in any chain from A to T. Hence, $a = \max(S, \succ^T)$.

Case 2: Suppose that S is not a prime. Consider $S \setminus \{x\}$. By Lemma 2, $S \setminus \{x\} \in \mathcal{M}$. Suppose that $S \setminus \{x\} \in \mathcal{P}$. If $S \setminus \{x\}$ is a prime atom, then $x = \max(S, \succ^{S\setminus\{x\}})$. If $S \setminus \{x\}$ is a prime that is not an atom, then by Lemma 6, there exists a unique prime chain connecting $S \setminus \{x\}$ to a prime atom $T \in \mathcal{P}$. Moreover, again by Lemma 6, the unique prime chain from $S \setminus \{x\}$ to T is included in any chain from A to T. Hence, $a = \max(S, \succ^T)$.

Suppose that $S \setminus \{x\} \notin \mathcal{P}$. Then, by Lemma 7, there exists $S^* \in \mathcal{M}$ such that $S \subsetneq S^*$ and $x \in C(S^*)$. Since $S \subsetneq S^*$, $|S^*| \geq k+1$. It follows from the induction hypothesis that $x = max(S^*, \succ^T)$ for some $T \in \mathcal{P}^{\mathcal{A}}$. Since $S \subsetneq S^*$, $x \in S$, and C satisfies *substitutability*, we obtain $x = max(S, \succ^T)$.

Part ii. Let C be an *acceptant* and *substitutable* choice rule. Suppose that C has an MC representation of size $m \in \mathbb{N}$, say for the priority profile $(\succ_1, \ldots, \succ_m)$. Consider the set of prime atoms $\mathcal{P}^{\mathcal{A}}$ of C. Consider a pair of distinct prime atoms $T, T' \in \mathcal{P}^{\mathcal{A}}$. Since T and T' are prime, there exist a unique $a \notin T$ and a unique $a' \notin T'$ such that $T \cup \{a\} \to T$ and $T' \cup \{a'\} \to T'$.

Now, we show that there exist $i, j \in \{1, \ldots, m\}$ such that $\succ_i \neq \succ_j$, and $a = max(T \cup \{a\}, \succ_i)$, $a' = max(T' \cup \{a'\}, \succ_j)$. By contradiction, suppose that there is a unique $k \in \{1, \ldots, m\}$ with $a = max(T \cup \{a\}, \succ_k)$ and $a' = max(T' \cup \{a'\}, \succ_k)$. Suppose, without loss of generality, that $a \succ_k a'$ or $a = a'$. Consider the choice set $T \cup T'$. Clearly, $a = max(T \cup T', \succ_k)$. Hence, $a \in C(T \cup T')$. Since $T \neq T'$ and C satisfies *substitutability*, there exists $x \notin T \cup \{a\}$ such that $a \in C(T \cup \{a, x\})$, which is a contradiction since $T \cup \{a\}$ is a parent of the prime T and by Lemma 4, a is no longer chosen whenever any other element is added to $T \cup \{a\}$. ■

Remark 1 First, we would like to note that it is not clear if a similar result can

be obtained for all path independent choice rules, in that one of our key observations, namely Lemma 5, does not hold in the absence of *acceptance*.[19] Second, let a choice rule C be maximizer-collecting* if there exists a priority profile $\pi = (\succ_1, \ldots, \succ_m)$ such that for each $S \in \mathcal{A}$, including the ones with $|S| \leq q$, $C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i)$. A choice set T is a prime* atom if $|T| = q - 1$, and there exists $a \notin T$ such that a is chosen from $T \cup \{a\}$, but no longer chosen whenever any other element is added to $T \cup \{a\}$. The only difference between a prime atom and prime* atom is that a prime atom has q elements, whereas a prime* atom has $q - 1$ elements. We claim that Theorem 1 holds with maximizer-collecting* choice rules and prime* atoms.[20]

3.1 Maximizer-collecting representation of responsive choice rules

A well-known example of a q-*acceptant* choice rule that satisfies *substitutability* is a responsive choice rule (see Example 1 for the definition). In Theorem 2, we show that the upper bound on the number of prime atoms is achieved by responsive choice rules. Put differently, responsive choice rules render an MC representation of the largest size among all *acceptant* and *substitutable* choice rules.

It is not difficult to see that the number of prime atoms of a responsive choice rule is $\binom{n-2}{q-1}$. Let C be a q-*acceptant* choice rule that is responsive to the priority ordering $\succ$. Let $a \in A$ be the $(n-1)^{th}$-ranked and b be the $(n)^{th}$-ranked element

[19] We presented a *path independent* choice rule that fails to satisfy Lemma 5 in Footnote 18.

[20] Our construction in the proof of Theorem 1 can be easily extended to obtain this result. To see this, consider the chains formed between the universal set of elements and prime atoms. As mentioned in the sketch of the proof, we obtain the priority orderings out of these chains. For each of these priority orderings, we are free to order the last q elements. For the extension, consider all the chains that connect the universal set of elements to prime* atoms. Similarly, the priority profile associated with these chains provides the canonical maximizer-collecting* representation. For this more stringent representation, one should additionally be careful about the $(n-q)^{th}$-ranked element at each priority. These longer chains discipline which elements are ranked at the $(n-q)^{th}$ position.

at $\succ$. Clearly, a set $S \in \mathcal{A}$ such that $|S| = q$ is a prime atom if and only if $a \notin S$ and $b \in S$. Since there are $\binom{n-2}{q-1}$ such sets, C has $\binom{n-2}{q-1}$ many prime atoms. The difficult part of the proof is to show that for an arbitrary *acceptant* and *substitutable* choice rule, the number of its prime atoms is at most $\binom{n-2}{q-1}$. To show this, we prove an important structural result, which states that for each *acceptant* and *substitutable* choice rule and for each $k \in \{q, \ldots, n\}$, there are exactly $\binom{n-k+q-1}{q-1}$ maximal choice sets with cardinality k (to emphasize, the number of maximal choice sets with a fixed cardinality is invariant among *acceptant* and *substitutable* choice rules). This structural result is formalized in Lemma 8. Next, we present and prove the result.

Theorem 2 *Each acceptant and substitutable choice rule C has an MC representation of a size less than or equal to $\binom{n-2}{q-1}$. If C is responsive, then C does not have an MC representation of any size smaller than $\binom{n-2}{q-1}$.*

To prove Theorem 2, we first prove a structural result in which we show that for each capacity q, universal set of elements with n members , and $k \in \{q, \ldots, n\}$, each *acceptant* and *substitutable* choice rule has the same number of maximal choice sets with k elements. We denote the collection of maximal choice sets with cardinality k by $\mathcal{M}_k$.

Lemma 8 *For each capacity q, and universal set of elements with n members, let C be an acceptant and substitutable choice rule. For each $k \in \{q, \ldots, n\}$, the number of maximal choice sets with cardinality k is $\binom{n-k+q-1}{q-1}$, i.e. $|\mathcal{M}_k| = \binom{n-k+q-1}{q-1}$.*

Proof. First, we argue that if for each $k \in \{q, \ldots, n\}$ the following identity holds, then we obtain the desired conclusion.

$$\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \sum_{i=k}^{n} \binom{i-q}{k-q} \binom{n-i+q-1}{q-1} \tag{2}$$

To see this, note that $|\mathcal{M}_n| = 1$ and for $k = n-1$, it follows from (2) that

$$\binom{n-1-q}{k-q} |\mathcal{M}_{n-1}| + \binom{n-q}{k-q} |\mathcal{M}_n| = \binom{n-1-q}{k-q} \binom{q}{q-1} + \binom{n-q}{k-q} \binom{q-1}{q-1}.$$

Since $|\mathcal{M}_n| = 1 = \binom{q-1}{q-1}$, we have $|\mathcal{M}_{n-1}| = \binom{q}{q-1}$. Similarly for $k = n-2$, we have $|\mathcal{M}_{n-2}| = \binom{q+1}{q-1}$. Proceeding inductively we obtain that $|\mathcal{M}_k| = \binom{n-k+q-1}{q-1}$. In

what follows we prove that (2) holds in two steps by showing that both sides of the equality are equal to $\binom{n}{k}$.

Step 1. We show that $\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \binom{n}{k}$. To see this, consider $\mathcal{K} = \{S \in \mathcal{A} : |S| = k\}$. Consider the partition of this set such that for each $S, S' \in \mathcal{K}$, S and S' belong the same part if and only if $C(S) = C(S')$. First, we show that

$$\mathcal{K} = \bigcup_{i=k}^{n} \bigcup_{S' \in \mathcal{M}_i} \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\} \tag{3}$$

Since for each $S \in \mathcal{A}$, there exists a unique $S' \in \mathcal{M}$ such that $C(S) = C(S')$ and $S \subset S'$, we get

$$\mathcal{K} = \bigcup_{S' \in \mathcal{M}} \{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} \tag{4}$$

Since $\{\mathcal{M}_i\}_{i=k}^{n}$ partitions $\{S' \in \mathcal{M} : |S'| \geq k\}$, we can rewrite (4) as

$$\mathcal{K} = \bigcup_{i=k}^{n} \bigcup_{S' \in \mathcal{M}_i} \{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} \tag{5}$$

Finally, note that for each $S' \in \mathcal{M}$ and $S \in \mathcal{K}$, if $C(S') \subset S \subset S'$, then *substitutability* implies that $C(S') = C(S)$. Therefore, $\{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} = \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}$. This observation together with (5) implies that (3) holds.

Now, if we count both sides of (3), then we obtain

$$\binom{n}{k} = \sum_{i=k}^{n} \sum_{S' \in \mathcal{M}_i} |\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}| \tag{6}$$

Next, we argue that for each $i \in \{k, \ldots, n\}$, and $S' \in \mathcal{M}_i$,

$$|\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}| = \binom{i-q}{k-q} \tag{7}$$

To see this, for each $i \in \{k, \ldots, n\}$, and $S' \in \mathcal{M}_i$, consider the set $\{T \subset S' \setminus C(S') : |T| = k - q\}$. Since $S' \in \mathcal{M}_i$, $|S' \setminus C(S')| = i - q$. It directly follows that $|\{T \subset S' \setminus C(S') : |T| = k - q\}| = \binom{i-q}{k-q}$. To show that (7) holds, we argue that $\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}$ is isomorphic[21] to $\{T \subset S' \setminus C(S') : |T| = k - q\}$. For each S that belongs to the former set, let $T = S \setminus C(S')$. Since $C(S') \subset S \subset S'$ and

[21] That is, there is a bijection between the two sets.

$|S| = k$, T belongs to the latter set. Conversely, for each T that belongs to the latter set, that is $T \subset S' \setminus C(S')$ with $|T| = k - q$, define $S = T \cup C(S')$. Since $|S| = k$ and $C(S') \subset S \subset S'$, S belongs to the former set. Thus, we obtain that (7) holds. Finally, if we combine (6) and (7), then we directly obtain that $\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \binom{n}{k}$.

Step 2. We show that $\sum_{i=k}^{n} \binom{i-q}{k-q}\binom{n-i+q-1}{q-1} = \binom{n}{k}$. To see this consider the set $\{1, 2, \ldots, n\}$ and let $\mathcal{K}$ be the collection of its subsets that contain k elements. Since there are $\binom{n}{k}$ such subsets, $|\mathcal{K}| = \binom{n}{k}$. Since $k > q$, for each $S \in \mathcal{K}$, there exists $q(S) \in S$ that is the qth highest number in S. Now, consider the partition of $\mathcal{K}$ such that for each $S, S' \in \mathcal{K}$, S and S' belong to the same part if and only if $q(S) = q(S')$. We denote this partition of $\mathcal{K}$ by $\mathcal{L}$. Now, note that for each $S \in \mathcal{K}$, $q(S) \in \{k+1-q, \ldots, n+1-q\}$. Next, for each $j \in \{k+1-q, \ldots, n+1-q\}$, we count the number of $S \in \mathcal{K}$ such that $q(S) = j$. If $q(S) = j$, then there $k - q$ numbers in S that are less than j, and $q - 1$ numbers that are greater than j. It follows that S can be chosen in $\binom{j-1}{k-q}\binom{n-j}{q-1}$ different ways. This observation together with $\mathcal{L}$ partitions $\mathcal{K}$ implies that

$$\binom{n}{k} = \sum_{j=k+1-q}^{n+1-q} \binom{j-1}{k-q}\binom{n-j}{q-1} \tag{8}$$

Finally, note that there are $n - k$ terms in this sum, and the first term is $\binom{k-q}{k-q}\binom{n-k+q-1}{q-1}$. Proceeding similarly yields that the right-hand side of (8) equals $\sum_{i=k}^{n} \binom{i-q}{k-q}\binom{n-i+q-1}{q-1}$. Thus, we obtain the desired equality. ■

Now, we are ready to prove Theorem 2, which prove by using Theorem 1 and Lemma 8.

Proof of Theorem 2. Let C be a *q-acceptant* and *substitutable* choice rule. Now consider the maximal choice sets with $q+1$ elements, i.e., $\mathcal{M}_{q+1} = \{S \in \mathcal{M} : |S| = q+1\}$. It follows from Lemma 5 that each $S \in \mathcal{M}_{q+1}$ has at most one prime child $S \setminus \{a\} \in \mathcal{P}^{\mathcal{A}}$. Therefore, the number of prime atoms is at most $|\mathcal{M}_{q+1}|$. By Lemma 8, $|\mathcal{M}_{q+1}| = \binom{n-2}{q-1}$. Thus, $\binom{n-2}{q-1}$ is an upper bound on the number of prime atoms. Hence, by Theorem 1, C has an MC representation of a size less than or equal to $\binom{n-2}{q-1}$.

Let C be a *q-acceptant* choice rule that is responsive to the priority ordering $\succ$.

We show that there exists $a \in A$ such that for each $S \in \mathcal{M}_{q+1}$, $S \setminus \{a\} \in \mathcal{P}^{\mathcal{A}}$. Let $a \in A$ be the $(n-1)^{th}$-ranked and b be the $(n)^{th}$-ranked element at $\succ$. First, note that since C is responsive to $\succ$, for each $S \in \mathcal{M}$ with $|S| = n-1$, the $q+1^{th}$-ranked element at $\succ$ is chosen from S. Proceeding similarly, we have for each $S \in \mathcal{M}_{q+1}$, $a \in C(S)$. Moreover, since C is responsive to $\succ$ and b is the bottom ranked element at $\succ$, for each $S, S' \in \mathcal{M}_{q+1}$, we have $S \setminus C(S) = S' \setminus C(S')$={b}.

We argue that for each $S \in \mathcal{M}_{q+1}$, $S \setminus \{a\}$ is prime. To see this, we use Lemma 4. For each $x \notin S \cup \{a\}$, consider $S \cup \{a, x\}$. Since $x \neq b$, we have $x \succ \{a\}$. It follows that $a \notin C(S \cup \{a, x\})$. Lemma 4 implies that $S \setminus \{x\}$ is prime. Now, since for each $S, S' \in \mathcal{M}_{q+1}$, $S \neq S'$, we have $S \setminus \{a\} \neq S' \setminus \{a\}$. That is each $S \in \mathcal{M}_{q+1}$ is the parent of a prime. This combined with Theorem 1 implies that C is m-MC if and only if $m = |\mathcal{M}_{q+1}|$. ■

Responsive choice rules are not unique in having the largest size canonical MC representation. To show this, in the next example, we construct a choice rule that is not responsive, but the number of its prime atoms equals $\binom{n-2}{q-1}$.

Example 4 Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider the priority profile $(\succ_\alpha, \succ_\beta, \succ_\gamma, \succ_\delta)$. Let C be the 2-*acceptant* choice rule that is MC of this priority profile.

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
1	1	2	2
3	2	3	3
4	4	4	1
5	5	5	6
6	6	6	5
2	3	1	4

The choice lattice $(\mathcal{M}, \searrow)$ associated with C is depicted in Figure 1.[22] It directly follows from the structure of this choice lattice and Lemma 4 that C has 4 prime atoms, namely $\{1, 6\}, \{4, 5\}, \{3, 6\}, \{2, 6\}$, which gives the upper bound on the number of

[22]Prime choice sets are colored by green.

prime atoms. Next, we argue that C is not responsive. By contradiction suppose C is responsive to a priority $\succ$. Now, since $C(\{2456\}) = \{2, 4\}$, 2 and 4 should be the two best $\succ$-ranked elements in this choice set. Let x be the next best ranked $\succ$-element in the same choice set. Now, we should have $C(\{256\}) \setminus \{2\} = C(\{456\}) \setminus \{4\} = x$. However, this is not the case, since when 4 is removed, 5 is chosen, but when 2 is removed, 6 is chosen.

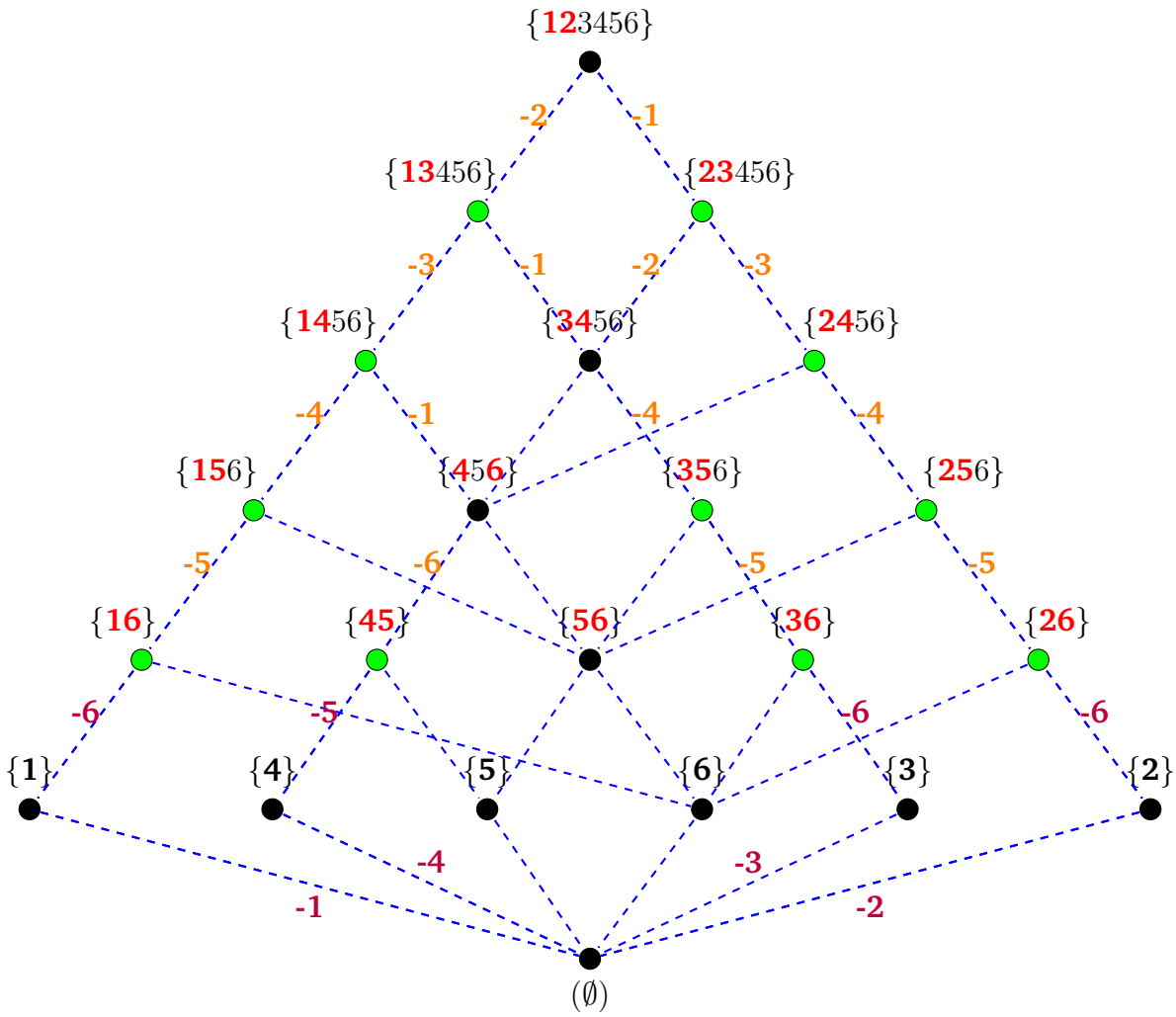

Figure 1: Lattice representation of the choice rule in Example 4

As a generalization, one can question if for each capacity constraint q and universal set of elements with n members, there exists an *acceptant* and *substitutable*

choice rule which is not responsive but renders a largest size MC-representation.[23] In our next result, we show that the answer is almost always in the affirmative.

Proposition 1 *For each capacity constraint $q \geq 2$ and universal set of elements with $n \geq q+2$ members, there exists an acceptant and substitutable choice rule that is not responsive, but has $\binom{n-2}{q-1}$-many prime atoms.*

Proof. For each $q \geq 2$ and $n \geq q+2$, let $A = \{a_1, \ldots, a_n\}$. Consider the priority ordering $a_n \succ a_{n-1} \succ \cdots \succ a_1$. Let $S' = \{a_n, \ldots, a_{n-q+1}\}$. Note that $|S'| = q-1$. Since $n \geq q+2$, $a_1, a_2, a_3 \notin S'$. Now, consider the choice rule C such that for each $S \in \mathcal{A}$, if $S = S' \cup \{a_1, a_2\}$, then $C(S) = S' \cup \{a_1\}$; otherwise, C is q-responsive with respect to $\succ$. Note that C is *q-acceptant* since $|S'| = q-1$ and $a_1 \notin S'$.

To see that C is *substitutable*, pick any $S_1, S_2 \in \mathcal{A}$ such that $S_1 \subsetneq S_2$. We show that $C(S_2) \cap S_1 \subset C(S_1)$. There are three cases to consider. Suppose that $S_2 = S' \cup \{a_1, a_2\}$. In this case, $C(S_2) = S' \cup \{a_1\}$. Since $S_1 \subsetneq S_2$ and $|S_2| = q+1$, $|S_1| \leq q$. Then, since C is q-responsive at S_1, $C(S_1) = S_1$. Thus, $C(S_2) \cap S_1 \subset C(S_1) = S_1$. Suppose that $S_2 \neq S' \cup \{a_1, a_2\}$ and $S_1 \neq S' \cup \{a_1, a_2\}$. In this case, C is q-responsive at S_1 and S_2. Hence $C(S_2) \cap S_1 \subset C(S_1)$. Suppose that $S_2 \neq S' \cup \{a_1, a_2\}$ and $S_1 = S' \cup \{a_1, a_2\}$. In this case, *substitutability* can be violated only if $a_2 \in C(S_2)$. To see that this is not possible, first note that $|S_2| \geq q+2$ since $S_1 \subsetneq S_2$. Moreover, since C is q-responsive at S_2 with respect to $\succ$ and a_2 is the second-worst element at $\succ$, we have $a_1, a_2 \notin C(S_2)$. Hence, C is *substitutable*.

Next, we argue that C is not responsive. Suppose that C is responsive to a priority ordering $\succ'$. Since $a_1 \in C(S' \cup \{a_1, a_2\})$ and $a_2 \notin C(S' \cup \{a_1, a_2\})$, we have $a_1 \succ' a_2$. Since $a_1 \notin C((S' \setminus \{a_n\}) \cup \{a_1, a_2, a_3\})$ and $a_2 \in C((S' \setminus \{a_n\}) \cup \{a_1, a_2, a_3\})$, we have $a_2 \succ' a_1$, which is a contradiction. Hence, C is not responsive.

Next, we argue that C is not responsive. First, note that if C is a responsive choice rule, then for each $S \in \mathcal{A}$ with $|S| \geq q+2$ and for each distinct pair $a, b \in C(S)$, we have $C(S \setminus \{a\}) \setminus C(S) = C(S \setminus \{b\}) \setminus C(S) = max(S \setminus C(S), \succ)$. Next, we show that C violates this property. Consider the choice set $S' \cup \{a_1, a_2, a_3\}$. We have

[23] We thank an anonymous referee for suggesting to consider this question.

$a_n, a_3 \in C(S' \cup \{a_1, a_2, a_3\})$. Note that $C(S' \cup \{a_1, a_2\}) = S' \cup \{a_1\}$ and $C(S' \cup \{a_1, a_3\}) = S' \cup \{a_3\}$. That is, if we remove a_n from $S' \cup \{a_1, a_2, a_3\}$, then a_2 is chosen; whereas if we remove a_3 from $S' \cup \{a_1, a_2, a_3\}$, then a_1 is chosen. Thus, we conclude that C is not responsive.

Finally, we argue that C has $\binom{n-2}{q-1}$-many prime atoms. To see this, first let $T = \{S \in \mathcal{A} : |S| = q-1, a_1, a_2 \notin S, \text{ and } S \neq S'\}$. Now, for each $S \in T$, $a_2 \in C(S \cup \{a_1, a_2\})$ and for each $a \notin S \cup \{a_1, a_2\}$, $a_2 \notin C(S \cup \{a_1, a_2, a\})$. Then, by Lemma 4, for each $S \in T$, $S \cup \{a_1\}$ is a prime atom. Note that $|T| = \binom{n-2}{q-1} - 1$. Next, we show that $S' \cup \{a_2\}$ is also a prime atom of C. To see this, consider $S' \cup \{a_1, a_2\}$. First note that $S' \cup \{a_1, a_2\} \in \mathcal{M}$, since it is the largest choice set in which a_1 is chosen. Also, $a_1 \in C(S' \cup \{a_1, a_2\})$, but for each $a \notin S' \cup \{a_1, a_2\}$, $a_1 \notin C(S' \cup \{a_1, a_2, a\})$. Then, by Lemma 4, $S' \cup \{a_2\}$ is also a prime atom. ■

4 Maximizer collecting choice rules of size q

Theorem 2 shows that the upper bound on the number of prime atoms is achieved by responsive choice rules. Hence, responsive choice rules have the largest size canonical MC representation. On the other hand, given capacity q, for each *acceptant* choice rule, the minimum number of priorities that can render an MC representation is at least q. In this section, we analyze the choice rules that are q-MC.

Choice rules that are q-MC are also lexicographic choice rules,[24] and they satisfy the following property. Let C be a choice rule that is lexicographic with respect to a list of priority orderings $(\succ_1, \ldots, \succ_q)$. We say that C is **immune to precedence effects** if for any permutation $\pi : \{1, \ldots, q\} \to \{1, \ldots, q\}$, C coincides with the choice rule C' that is lexicographic with respect to $(\succ_{\pi(1)}, \ldots, \succ_{\pi(q)})$. Note that, if a choice rule C is q-MC of a priority profile $(\succ_1, \ldots, \succ_q)$, then C is also lexicographic with respect to $(\succ_1, \ldots, \succ_q)$ and immune to precedence effects.

In what follows, we first characterize *q-acceptant* choice rules that are q-MC. It

[24] See Example 2 for a formal definition of a lexicographic choice rule.

follows from this characterization that if the difference between the size of the universal set of elements and the capacity is bigger than two, then there is no q-*acceptant* choice rule that is q-MC. Although a lexicographic choice rule that is immune to precedence effects does not necessarily have to be q-MC,[25] this impossibility result still provides a sense in which avoiding such precedence effects is difficult.

To characterize *acceptant* choice rules that are q-MC, we introduce a new property called *strong blocking*. Given a choice rule C, an element a **blocks** another element b in a choice set S if a is chosen in S and b is not chosen in S, but b is chosen when a is removed from S, i.e. if $a \in C(S), b \notin C(S)$, but $b \in C(S \setminus \{a\})$.

Strong blocking requires that if an element a blocks another in a choice set S that contains more than $q+1$ elements, then a continues to block b in any subset of S that contains more than $q+1$ elements including a and b. Formally, a choice rule satisfies **strong blocking** if the following is satisfied: for each choice set S with $|S| > q+1$, if an element a blocks another element b in S, then for each $S' \subset S$ with $a, b \in S'$ and $|S'| > q+1$, a blocks b in S'. Proposition 2 characterizes q-*acceptant* choice rules that are q-MC.

Proposition 2 *Let C be a q-acceptant choice rule. C is q-MC if and only if C satisfies substitutability and strong blocking.*

For the *if part* of the result, we provide two different proofs. First proof is direct, self-contained, and provides an explicit construction of the priority profile. Our second proof is shorter, but uses the terminology and results from Theorem 1's proof. Since these proofs provide different insights for the result, we present both of them.

Proof. (*Only if part*) Consider a q-*acceptant* choice rule C that is q-MC of the priority profile $(\succ_1, \ldots, \succ_q)$. It is easy to see that C is *substitutable*. To see that C satisfies *strong blocking*, let $a, b \in A$ be such that a blocks b. Then, there exists $S \in \mathcal{A}$ and $i \in \{1, \ldots, q\}$ such that $a \succ_i b$ and for each $c \in S \setminus \{a, b\}$, $b \succ_i c$. Now, let $S' \subset S$ be

[25] For example, each responsive choice rule is lexicographic and immune to precedence effects, but not necessarily q-MC.

such that $a \in S'$ and $|S'| > q+1$. Consider the set $S'' = S' \setminus \{a\}$. Since $|S''| \geq q+1$ and b is top ranked by $\succ_i$ in S'', $b \in C(S'')$. Since C must choose q distinct elements from S'', there cannot be any other priority that top ranks b in S''. Since $a \succ_i b$, there is no priority that top ranks b in S'. Thus, $b \notin C(S')$.

(*If part*) **First proof:** Let C be a *q-acceptant* choice rule that satisfies *substitutability* and *strong blocking*. First, we construct a priority profile $(\succ_1, \ldots, \succ_q)$. Since C is *q-acceptant*, $|C(A)| = q$. Let $C(A) = \{a_{11}, \ldots, a_{q1}\}$. For each $i \in \{1, \ldots, q\}$, let $a_{i2} = C(A \setminus \{a_{i1}\})$. Similarly, for each $j \in \{2, \ldots, n-q+1\}$, let $a_{ij} = C(A \setminus \{a_{i1}, \ldots a_{ij-1}\}) \setminus C(A)$. Note that since C satisfies *substitutability*, $C(A) \setminus \{a_{i1}\} \subset C(A \setminus \{a_{i1}, \ldots a_{ij-1}\})$. Since C satisfies *q-acceptance*, $C(A \setminus \{a_{i1}, \ldots a_{ij-1}\}) \setminus C(A)$ is a singleton. Therefore, for each $j \in \{2, \ldots, n-q+1\}$, a_{ij} is well-defined. Now, for each $i \in \{1, \ldots, q\}$, define $\succ_i$ such that $a_{i1} \succ_i a_{i2} \cdots \succ_i a_{in-q+1}$. Note that we did not specify how $\succ_i$ ranks the elements in $C(A) \setminus \{a_{i1}\}$. Let $\succ_i$ rank these elements at the bottom such that $a_{in-q+1} \succ_i a_{11} \cdots a_{(i-1)1} \succ_i a_{(i+1)1} \cdots \succ_i a_{q1}$.

First we show that for each $S \in \mathcal{A}$ and $a \in A$, if $a = max(S, \succ_i)$ for some $i \in \{1, \ldots, q\}$, then $a \in C(S)$. To see this, suppose that a is the jth-ranked element in $\succ_i$, i.e. $a = a_{ij}$. Since $a_{ij} = max(S, \succ_i)$, for each $k \in \{1, \ldots, j-1\}$, $a_{ik} \notin S$. It follows that $S \subset A \setminus \{a_{i1}, \ldots a_{ij-1}\}$. Now, since C satisfies *substitutability* and $a_{ij} = C(A \setminus \{a_{i1}, \ldots a_{ij-1}\})$, $a \in C(S)$.

In what follows we show that for each $S \in \mathcal{A}$ with at least $q+1$ elements, if $a \in C(S)$, then $a = max(S, \succ_i)$ for some $i \in \{1, \ldots, q\}$. By contradiction, suppose there exists such $S \in \mathcal{A}$ for which this is not true. Next, let $\mathcal{F}$ be the collection of all such choice sets, i.e. $\mathcal{F} = \{S \in \mathcal{A} : |S| \geq q+1 \text{ and } C(S) \neq \cup_{i \in \{1,\ldots,q\}} max(S, \succ_i)$. We suppose that $\mathcal{F} \neq \emptyset$. Let $S^* \in \mathcal{F}$ be a maximal set according to set containment, i.e. there is no $S \in \mathcal{F}$ such that $S^* \subsetneq S$. Next, we make two key observations about S^*.

Since $S^* \in \mathcal{F}$ and $C(S^*) \subset \cup_{i \in \{1,\ldots,q\}} max(S^*, \succ_i)$, there exists $a \in S^*$ such that for some distinct $i, j \in \{1, \ldots, q\}$, $a = max(S^*, \succ_i)$ and $a = max(S^*, \succ_j)$. First, we show that there can not be a third distinct priority $\succ_k$ with $a \in max(S^*, \succ_k)$. To see this we proceed by contradiction. Since for each priority, there is a distinct top ranked

element at A, a can be top ranked by at most one of the priorities $\{\succ_i, \succ_j, \succ_k\}$ at A. Let x be top ranked by one of the other priorities at A, and consider the set $S^* \cup \{x\}$. Now, note that either a or x must be top ranked by two of the priorities $\{\succ_i, \succ_j, \succ_k\}$ at $S^* \cup \{x\}$, which implies $S^* \cup \{x\} \in \mathcal{F}$. This contradicts that S^* is maximal in $\mathcal{F}$.

It follows from our previous observation that there are exactly two priorities $\succ_i$ and $\succ_j$ that top ranks a at S^*. Next, we observe that $a \neq max(A, \succ_i)$ and $a \neq max(A, \succ_j)$. By contradiction, suppose w.l.o.g. that $a = max(A, \succ_i)$. Recall that by our construction of $\succ$, for each $\succ_i$, $\{a_{1k}\}_{k \neq i}$ are bottom ranked by $\succ_i$. Now, since $a = max(A, \succ_i)$, we get $S^* \subset \{a_{11}, \ldots, a_{q1}\}$. This contradicts that $|S^*| \geq q + 1$.

Since a is not top ranked by $\succ_i$ or $\succ_j$ at A, consider a_{i1} and a_{j1}, which are different from a and each other. Next, consider the set $S^* \cup \{a_{i1}, a_{j1}\}$. We show that a_{i1} blocks a at $S^* \cup \{a_{i1}, a_{j1}\}$. First, note that since $a_{i1} = max(S^* \cup \{a_{i1}, a_{j1}\}, \succ_i)$, $a_{i1} \in C(S^* \cup \{a_{i1}, a_{j1}\})$. However, since a is top ranked only by $\succ_i$ and $\succ_j$ at S^*, there is no priority that top ranks a at $S^* \cup \{a_{i1}, a_{j1}\}$. Now, note that $a \notin C(S^* \cup \{a_{i1}, a_{j1}\})$, if not, then $S^* \cup \{a_{i1}, a_{j1}\} \in \mathcal{F}$, contradicting that S^* is maximal in $\mathcal{F}$. Moreover, since $\succ_i$ top ranks a at $S^* \cup \{a_{j1}\}$, $a \in C(S^* \cup \{a_{j1}\})$. It follows that a_{i1} blocks a at $S^* \cup \{a_{i1}, a_{j1}\}$. However, we also have $a \in C(S^* \cup \{a_{i1}\})$, since $\succ_j$ top ranks a at $S^* \cup \{a_{i1}\}$. Since $|S^* \cup \{a_{i1}\}| > q + 1$, this violates *strong blocking*. Thus we conclude that $\mathcal{F} = \emptyset$. It follows that for each $S \in \mathcal{A}$, we have $C(S) = \cup_{i \in \{1, \ldots, q\}} max(S, \succ_i)$.

Second proof: Let C be a *q-acceptant* and *substitutable* choice rule that satisfies *strong blocking*. It follows from Theorem 1 that C is is m-MC, where m is the number of prime atoms of C. First note that $m \geq q$. To see this, note that for each $a \in C(A)$, $A \setminus \{a\}$ is a prime choice set. Moreover, Lemma 6 implies for each $a \in C(A)$, we obtain a prime chain (a chain consisting of primes only) starting with A, including $A \setminus \{a\}$, and ending up at a distinct prime atom T^a. Moreover, by Lemma 5, each prime that is not an atom must have a unique prime child. Therefore, these are the only prime chains that connects A to a prime atom. Now, by contradiction suppose that C is not q-MC, i.e. $m > q$. It follows from the previous observation that there exists a prime atom T such that there is no prime chain that connects A to T.

Now, consider this prime atom T. We show that there exists a prime choice set S such that $T \subset S$ and none of the parents of S is a prime. If not, then consider any choice set S_1 that is a parent of T. It follows that S_1 is a prime and all parents of S_1 are primes as well. Then, by proceeding similarly we obtain a prime chain that connects A to T. This contradicts the choice of T.

Next, consider this prime choice set S such that $T \subset S$ and none of the parents of S is a prime. Since each maximal choice set with $n-1$ elements is prime, it follows that $|S| < n-2$. Since S is a prime, let $S \cup \{a\}$ be the unique parent of S. Since $S \cup \{a\}$ is not prime, it should have at least two parents $S \cup \{a, b\}$ and $S \cup \{a, c\}$.

Next, we show that $C(S \cup \{a, b\}) \cap S = C(S \cup \{a, c\}) \cap S$. It follows from Lemma 4 that $a \notin C(S \cup \{a, b\})$ and $a \notin C(S \cup \{a, c\})$. Then, since C is *substitutable*, $C(S \cup \{a, b\}) \setminus \{b\} = C(S \cup \{a\}) \setminus \{a\} = C(S \cup \{a, c\}) \setminus \{c\}$. Since $a \notin C(S \cup \{a, b\})$ and $a \notin C(S \cup \{a, c\})$, it follows that $C(S \cup \{a, b\}) \cap S = C(S \cup \{a, c\}) \cap S$.

Finally, consider $S \cup \{a, b, c\}$. Since $S \cup \{a, b\}$ and $S \cup \{a, c\}$ are maximal sets, $b, c \in C(S \cup \{a, b, c\})$. Let $\{d\} = C(S \cup \{a, b\}) \setminus C(S \cup \{a, b, c\})$. Since in both choice sets b is chosen and a is not, we have $d \in S$. It follows that c blocks d in $S \cup \{a, b, c\}$. However, since $C(S \cup \{a, b\}) \cap S = C(S \cup \{a, c\}) \cap S$, we also have $d \in C(S \cup \{a, c\})$. This contradicts that C satisfies *strong blocking*. ■

Theorem 3 shows that if the difference between the size of the universal set of elements and the capacity is bigger than two, then there is no q-*acceptant* choice rule that is q-MC.

Theorem 3 *For each capacity constraint q and universal set of elements with n members, if $q > 3$ and $n > q + 2$, then there is no q-acceptant choice rule that is q-MC.*

Proof. We use Proposition 2 and show that if $q > 3$ and $n > q + 2$, then there is no q-*acceptant* choice rule C that satisfies *substitutability* and *strong blocking*. To see this, first let $C(A) = \{a_1, a_2, \ldots, a_q\}$. Note that since $n > q+2$, there are at least three distinct elements $\{b_1, b_2, b_3\}$ such that for each $i \in \{1, 2, 3\}$, $b_i = C(A \setminus \{a_i\}) \setminus C(A)$. Now, consider the choice set $S = C(A) \cup \{b_1, b_2, b_3\}$. Since C satisfies *substitutability*,

$C(S) = C(A)$. Moreover, since $q > 3$, there exists $a_4 \in C(A) \setminus \{a_1, a_2, a_3\}$. Next, consider the choice set $S \setminus \{a_4\}$. Since C is *q-acceptant*, $C(S) \cap \{b_1, b_2, b_3\} \neq \emptyset$. Suppose w.l.o.g that $b_1 \in C(S \setminus \{a_4\})$. It follows that a_4 blocks b_1 at S. Now, consider the choice set $S \setminus \{a_1\}$. We have $S \setminus \{a_1\} \subset S$ and $|S \setminus \{a_1\}| > q + 1$. But, since $b_1 \in C(A \setminus \{a_1\})$, *substitutability* implies that $b_1 \in C(S \setminus \{a_1\})$. Thus, a_4 fails to block b_1 at $S \setminus \{a_1\}$, indicating that C violates *strong blocking*. Therefore, it follows from Proposition 2 that if $q > 3$ and $n > q + 2$, then there is no *q-acceptant* choice rule C that is q-MC. ■

If $n = q + 2$ or $n = q + 1$, then there exist *q-acceptant* and q-MC choice rules. To see this, let $A = \{a_1, \ldots, a_q, x, y\}$. Consider a priority profile $\pi = (\succ_1, \ldots, \succ_q)$ such that for each $i \in \{1, \ldots, q\}$, $a_i = max(A, \succ_i)$, and at each priority $\succ_i$, x is second ranked. Now, consider the choice rule C that is MC of π. It directly follows that for each choice set S with $q+1$ elements, C chooses q-many distinct elements. It follows that C is *q-acceptant*.

A related question is as follows. Consider all the q-acceptant and substitutable choice rules. Let q^* be the minimum number of priorities needed to represent a choice rule from this family. Put differently, q^* is the minimum number of the prime atoms that a choice rule in this family has. It follows from our Theorem 3 that if $q > 3$ and $n > q + 2$, then $q^* > q$. However, we are silent about what is the q^* value for each given capacity q, and leave this as an open question.

5 Conclusion

We have analyzed *acceptant* and *substitutable* choice rules. Despite all their eminence for economic applications, *acceptant* and *substitutable* choice rules had lacked a canonical representation. More precisely, the size of a smallest size MC representation of an *acceptant* and *substitutable* choice rule and how to construct such a representation had been unknown. We have addressed this problem by introducing the notion of a prime atom and constructively proving that for each *acceptant* and

substitutable choice rule, the number of its prime atoms determines the smallest size maximizer collecting representation.

Our representation result is relevant for economic applications in which *acceptant* and *substitutable* choice rules are used. A prominent example is the school choice problem, in which each school specifies its admissions policy in the form of an *acceptant* and *substitutable* choice rule that reconciles the objective of admitting students with high exam scores together with some other policies such as affirmative action or diversity. A natural question in this case is whether there is a simple way to communicate to the public how these choice rules operate. Since such a choice rule has an MC representation, it is natural to assume that as the size of this representation decreases, the communication can be easier. Our Theorem 1 and the related construction in its proof shows how to choose the priority orderings as to obtain a minimal size MC representation.

We have also analyzed q-MC choice rules that are both easy to communicate and immune to precedence effects highlighted by Kominers and Sönmez (2016) and Dur et al. (2013). These features of q-MC choice rules make them particularly appealing for diversity-motivated designs, such as affirmative action policies implemented in school choice. We have characterized *acceptant* choice rules that are q-MC. An impossibility result follows from this characterization, indicating that if the difference between the size of the universal set of elements and the capacity is bigger than two, then there is no q-MC choice rule. This result provides a sense in which such precedence effects may be unavoidable.

In summary, the main contributions of our study are to provide a canonical MC representation for each *acceptant* and *substitutable* choice rule, and to discuss some implications for school choice applications. Although our study provides an extensive treatment of the subject, it also leads to a variety of open problems yet to be solved.

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