Choice Regularities *

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Abstract

This paper is an attempt to investigate boundedly rational choice theories based on the structure of the choice axioms that these theories satisfy. We propose *choice regularities* to classify choice theories according to the syntactic structure of the underlying choice axioms. Based on choice regularities, we find the class of choice theories rendering characterizations that are as simple as that of rational choice theory. Then, we propose and illustrate a method to identify the simple behavioral differences that distinguish a choice theory in a given family of choice theories.

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1 Introduction

Samuelson's classical result (Samuelson (1938)) characterizes the rational choice theory via a simple condition, the *weak axiom of revealed preference* (WARP). WARP requires that *if* an alternative is chosen from a set of alternatives, *then* from each subset of this set, the same alternative should be chosen whenever it is available. The simplicity of WARP provides normative appeal and ease of identification from the observables for rational choice theory. Recent and growing literature on boundedly rational choice proposes plausible choice theories to accommodate choice behavior that rational choice theory fails to explain. The richness of human choice behavior calls for the classification of the choice theories based on the properties that these theories satisfy. The commonly followed methodology is to characterize the choice theories based on general principles called *choice axioms*. Although axiomatic characterizations provide classifications of choice theories and bring observable content to each choice theory, several questions remain untouched: *Is it possible to analyze the structure of a choice axiom in formal terms? Which boundedly rational choice theories render characterizations that are as simple as WARP? How can we identify simple behavioral differences that distinguish a choice theory from the others?* This paper is an attempt to answer these three questions.

In the classical revealed preference framework, the observable content of a decision maker's choices is summarized by a *choice function*, which singles out an alternative from each *choice set*.[1] In this framework, a *choice theory* is a collection of choice functions. For example, rational choice theory is the collection of choice functions that can be represented as if maximizing a single preference relation. Similarly, a choice axiom proposes a requirement to be satisfied by choice functions. That is, a choice axiom is associated with the choice theory consisting of the choice functions that satisfy the axiom. We use this association between choice axioms and choice

[1] A choice set is a subset of the grand alternative set with at least two alternatives. We read $a = c(S)$ as alternative a is chosen from the choice set S.

theories to explore the structure of choice axioms.

Several studies analyze the *computational complexity* of decision-making, which measures the time or effort needed for the empirical refutation of a choice theory.[2] Depending on the proposed measure, some of these studies favors boundedly rational choice theories, while others find the rational choice theory computationally more tractable. Our aim in here is not to introduce a new computational complexity measure, in contrast, we aim to introduce a *descriptive measure* for analyzing the structure of choice axioms which paves the way for the axiomatic classification and comparison of choice theories.[3]

A notion that would discipline the structure of a choice axiom should be both precise and permissive to accommodate the rich human choice behavior. From among several plausible formalizations, we propose *choice regularities* to classify choice theories according to the structure of the underlying choice axioms. We believe that the justification for adopting choice regularities will reside in their implications for the identification of choice theories. One of our aims in here is to illustrate the usefulness of choice regularities by addressing the second and third questions that we pose.

For a given choice function c, a choice regularity is an "*if* ... *then*" formula. We consider two specifications of these formulas: *the free regularities* and *the universal regularities*. Throughout the paper, we use the generic term *regularity* whenever it applies both for a free or universal regularity. In particular, a *first order free regularity* is a formula of the form:

$$\textit{If } a_1 = c(S_1), \textit{ then } a_2 = c(S_2)$$

for some pair of alternatives a_1, a_2 and choice sets S_1, S_2. In general, a k^{th}*-order free*

[2]As an incomplete list see Futia (1977), Johnson (2006), Apesteguia & Ballester (2010), Demuynck (2010), Salant (2011), Mandler (2015). We discuss the relation to these studies in Section 3 and 5.

[3]A branch of *computational complexity theory* and of *finite model theory*, namely, *descriptive complexity theory* investigates the relationship between the computational complexity and logic. We discuss the relation in Section 2.

regularity (k-*f*reg) is a formula of the form:

$$\textit{If } a_1 = c(S_1) \textit{ and } a_2 = c(S_2) \cdots \textit{ and } a_k = c(S_k), \textit{ then } a_{k+1} = c(S_{k+1})$$

for a set of alternatives $a_1, \ldots, a_{k+1}$ and choice sets $S_1, \ldots S_{k+1}$.[4]

A choice theory is k^{th}*-order free regular* (*k-fregular*) if a collection of k^{th}-order free regularities identifies the theory. That is, a choice function belongs to the theory if and only if the choice function satisfies all of the listed k^{th}-order free regularities. For example, choice axioms that are as simple as WARP are representable as first order free regularities. To see this, recall that WARP requires if $a = c(S_1)$, then $a = c(S_2)$ for each alternative a and for each pair of choice sets S_1, S_2 such that S_2 is a subset of S_1. Since WARP is representable as a collection of first order free regularities, it is a 1-*f*regular choice axiom. Equivalently, rational choice theory is a 1-*f*regular choice theory.

A free regularity leaves the logical relationships between the *variables* of the formula, namely the alternatives and the choice sets, unspecified. Therefore, a single choice axiom, such as WARP, is represented as a collection of free regularities, instead of a single one. Although this freedom brings generality for identifying choice theories,[5] which is crucial for their classification, it lacks precision in capturing the structure of a choice axiom. As to complement the analysis of free regularities, we introduce and analyze *universal regularities*. A universal regularity is an "*if* ... *then*" formula in which all the variables are *universally quantified*. That is, a universal regularity requires the "*if* ... *then*" formula to be satisfied for each variable that appears in the formula. To specify universal regularities we confine ourselves to three set of variables, namely, *alternatives*, *choice sets*, and *nested pairs of choice sets*. A choice theory is k^{th}*-order universally regular* (k-*u*regular) if a collection of k^{th}-order univer-

[4]The formulation of regularities is related to Mathematical Logic, in that a regularity can be represented as a *formula* in a properly defined *language* of propositional calculus. For the formal definitions see Section 2. I am grateful to Ariel Rubinstein for pointing out the connection to Mathematical Logic.

[5]In this vein, in Proposition **??** we show that this freedom makes it possible to identify each choice theory as a k-fregular theory for some k, and provide a uniform upper bound on k.

sal regularities identifies the theory. Since all the variables in WARP are universally quantified, WARP is a single 1-*u*regularity. Therefore, rational choice theory is also 1-*u*regular.

Next, we use the formalism of choice regularities to address the second question that we pose in this paper: Which boundedly rational choice theories render characterizations that are as simple as WARP? To answer this question, observing that most of the boundedly rational choice theories share two common features, namely they are *neutral*[6] and nest rational choice, in Proposition 1, we show that a neutral choice theory that nests rational choice is 1-*f*regular if and only if the choice theory is *category wise rational*. We also show that rational choice theory is the only 1-*u*regular choice theory in this class.

To illustrate our category wise rationality, let us revisit Luce and Raiffa's Dinner (Luce & Raiffa (1957)) in which they choose chicken when the menu consists of steak and chicken only, yet go for the steak when the menu consists of steak, chicken, and frog's legs. More generally, presence of a specific alternative a^* (frog's leg) signals about the quality of the other items in the menu.[7] Consequently, the decision maker categorizes the menus into two: those that contain a^*, and those that do not. In the former case he maximizes a preference relation $\succeq_1$, and in the latter he maximizes another preference relation $\succeq_2$.

In general, the primitive of a category wise rational choice theory is a *categorization* $\mathcal{P}$ of the choice sets, which is a collection of *categories* $\{\Omega_i\}_{i\in I}$ such that each choice set is contained by at least one category in this collection, and for each $i \in I$, category Ω_i is *closed under union*, that is, if choice sets S and S' are in Ω_i, then the union of S and S' is in Ω_i.[8] A choice function c is *category wise rational with respect to*

[6]A choice theory is neutral if for each choice function that belongs to the choice theory, any choice function that exhibits the same choice behavior for a relabeling of the alternatives also belongs to the given choice theory.

[7]Sen (1993) names this phenomenon as "epistemic value of the menu".

[8]In psychology, categorization is also taken as a central concept for human decision making and

a categorization $\mathcal{P}$ of the choice space, if there is a collection of preference relations $\{\succeq_i\}_{i \in I}$ such that for each choice set S and $i \in I$, if S belongs to category Ω_i, then $c(S)$ is the $\succeq_i$-maximal alternative in S.[9]

In choice theory literature, many studies provide axiomatic characterizations of choice theories. A basic problem that is often encountered is to illustrate the behavioral differences of a specific choice theory from the others that have similar representations or accommodate the same documented choice behavior. Since choice regularities, free or universal, provide a plausible syntactic classification of choice axioms, we are endowed with a natural structure to constrain the choice axioms for ease of description. Thereof, in the second part of the paper, we propose the *relative identification* of choice theories by using axioms that are constrained to be representable as *second order regularities*.[10]

To illustrate the relative identification method, we consider a specific family of boundedly rational choice theories. Namely, Rationalization via Game Trees (Xu & Zhou (2007)), Sequentially Rational Choice with Binary Rationales (Manzini & Mariotti (2007), Apesteguia & Ballester (2013)), Choice with Limited Attention (Masatlioglu et al. (2012)), and List-Rational Choice (Yildiz (2016)). First, we find out all the second order free regularities that each theory satisfies. We observe that these free regularities can be represented in the form of a set of universal regularities. It follows that this family of choice theories can be relatively identified via second order regularities that are both free and universal.

The second order regular choice axioms that account for the simple behavioral differences might be difficult to infer from the characterization results. For exam-

analyzed extensively. For a comprehensive reference see Cohen & Lefebvre (2005).

[9]Category wise rationality is a restricted version of *rationalizability by multiple rationales* (Kalai et al. (2002)). We explore this connection in Section 3.

[10]One can question if we can adopt first order regularities for relative identification. Note that any 1-regularity can also be expressed as a 2-regularity. However, boundedly rational choice theories that accommodate well-documented menu dependencies, such as the attraction effect, compromise effect, or recency effect, do not satisfy any nontrivial first order regularity.

ple, our *weak path independence* axiom requires if an alternative a is chosen from a choice set, and b is chosen when a is removed from the set, then a should be chosen when compared to b. A choice function that satisfies all the axioms that we propose in Section 4 but weak path independence can not be consistent with sequentially rational choice with binary rationales, but it can be consistent with rationalization via game trees. Put differently, for the comparison between these two theories, if we restrict ourselves to second order regular axioms, the entire behavioral difference is captured by weak path independence. Similarly, via our results in Section 4, we aim to illustrate that relative identification nicely complements the axiomatic characterizations of the choice theories by highlighting the simple behavioral differences that distinguish each theory from the others.

2 Choice Regularities

Let A be a fixed nonempty finite alternative set with n alternatives. Let Ω denote the collection of all subsets of A with at least two alternatives. Let $\mathcal{N}$ stand for the pairs of nested choice sets, that is $\mathcal{N} = \{(S_i, S_j) | S_i, S_j \in \Omega$ such that $S_i \subset S_j\}$. A choice function is a mapping $c : \Omega \rightarrow A$ such that for each $S \in \Omega$, $c(S) \in S$.

A choice theory τ is a collection of choice functions. We consider two choice procedures with possibly different formulations as equivalent if these procedures are observationally indistinguishable in the revealed preference framework. That is, two choice procedures rationalize the same set of choice functions. Similarly, we associate a choice axiom with the choice functions that satisfy the axiom, and use this association to explore the structure of a choice axiom.

We introduce few basic concepts from Mathematical Logic to define our regularity notions as *formulas* in a properly defined *language* of propositional calculus.[11] Here we consider a specific language $\mathcal{L}$ with the following alphabet:

[11] For more on Mathematical Logic one can consult Crossley et al. (2012)

1. Variables: ”alternatives” $(a_1, \ldots, a_n)$, ”choice sets” $(S_1, \ldots S_m)$, ”nested pairs of choice sets” $((S_1, S_2), \ldots, (S_k, S_l))$.

2. Logical connectives: conjunction $(\wedge)$, implication $(\Rightarrow)$.

3. Quantifiers: the universal quantifier, ”for each” $(\forall)$.

4. Atomic statement (predicate): ”a is chosen from S” $(a = c(S))$.

For a given language, such as $\mathcal{L}$, a **formula** in this language is constructed from the atomic statements by using the logical connectives. Let q be a formula in a given language. A variable v is **universal** in q if ”$\forall v$” appears in q. Otherwise it is said to be **free**. A formula is a **sentence** if it contains no free variables. For the given language $\mathcal{L}$ that is specified above, for each choice function c, **a first order free regularity** (**1-*f*reg**) is a formula of the form:

$$\text{If } a_1 = c(S_1), \text{ then } a_2 = c(S_2)$$

for some $a_1, a_2 \in A$ and $S_1, S_2 \in \Omega$. Similarly, a k^{th}**-order free regularity (k-*f*reg)** is a formula of the form:

$$\text{If } a_1 = c(S_1) \text{ and } a_2 = c(S_2) \cdots \text{ and } a_k = c(S_k), \text{ then } a_{k+1} = c(S_{k+1}).$$

for some $a_1, \ldots, a_{k+1} \in A$ and $S_1, \ldots S_{k+1} \in \Omega$.

For each choice function c, **a first order universal regularity (1-*u*reg)** is a sentence of the language $\mathcal{L}$ which has the following form:

$$\text{If } a_1 = c(S_1), \text{ then } a_2 = c(S_2)$$

for each $a_1, a_2 \in A$, and **for each** $S_1, S_2 \in \Omega$ or **for each** $(S_i, S_j) \in \mathcal{N}$ where $i, j \in \{1, 2\}$. Similarly, a k^{th}**-order universal regularity (k-*u*reg)** q is a *sentence* of the language $\mathcal{L}$ such that q is an “*if* ... *then*” requirement in which the precedent part contains k many atomic statements and the consequent part contains a single atomic statement. Since q is required to be a sentence of the language $\mathcal{L}$, all the variables that are used in q are universal . Put differently, a k^{th}-order universal regularity is **the universal closure** of a k^{th}-order free regularity, that is obtained by adding a universal quantifier for every free variable of the k^{th}-order free regularity.

We use the term **regularity** when it applies both for a free or universal regularity. A choice theory τ **satisfies a set of regularities** Q if each $c \in \tau$ satisfies each regularity $q \in Q$. Now, we are ready to provide the main definitions used to classify the choice theories based on the structure of the underlying choice axioms.

Definition. *Let τ be a choice theory.*

> *τ is k^{th}**-order free regular (k-fregular)** if a collection of k-fregs Q^f identifies τ, i.e. a choice function c satisfies each $q^f \in Q^f$ if and only if $c \in \tau$.*
>
> *τ is k^{th}**-order universally regular (k-uregular)** if a collection of k-uregs Q^u identifies τ, i.e. a choice function c satisfies each $q \in Q^u$ if and only if $c \in \tau$.*

Since each k^{th}-order universal regularity can be equivalently represented as a collection of k^{th}-order free regularities, each k^{th}-order universally regular choice theory is also k^{th}-order free regular.

Comments on the choice of the formal language:

We choose the language $\mathcal{L}$ to analyze the syntax of choice axioms. One can consider alternative languages for the same purpose. For example, consider the language $\mathcal{L}'$ obtained by adding "the negation" ($\neg$) and "or" ($\vee$) to the logical connectives of $\mathcal{L}$. In this extended language $\mathcal{L}'$ we can additionally write formulas of the form: If $a_1 = c(S_1)$ and $a_2 \neq c(S_2)$, then $a_3 = c(S_3)$ or $a_4 = c(S_4)$ for some $a_1, a_2, a_3, a_4 \in A$ and $S_1, S_2, S_3, S_4 \in \Omega$. However, it is straightforward to observe that for each formula q in $\mathcal{L}'$, there is a set of formulas Q in $\mathcal{L}$ such that a choice function c satisfies q if and only if c satisfies Q. Put differently, any choice behavior that can be identified in $\mathcal{L}'$ can also be identified in the more primitive language $\mathcal{L}$. This is one reason for confining ourselves to the language $\mathcal{L}$.

We also allow set operations ”union” and ”set difference” to appear within the formulas we use. Alternatively, one can eliminate these and instead use a richer set of variables including pairs of alternative-choice set pairs (a, S) such that $a \in S$ or $a \notin S$.

3 First order regular choice theories

Most of the boundedly rational choice theories analyzed in the literature share two common features. First, these theories nest rational choice. Second, these theories are *neutral*, that is, if a choice function belongs to a choice theory, then any choice function that exhibits the same choice behavior for a relabeling of the alternatives also belongs to the given choice theory. Next, we formally define the neutrality of a choice theory.

Definition. *Let c be a choice function and $\pi : A \to A$ be a bijection on A, c^π is the choice function such that for each $S \in \Omega$, $c^\pi(S) = \pi(c(\pi(S)))$. A choice theory τ is* ***neutral*** *if for each $c \in \tau$ and bijection $\pi : A \to A$, we have $c^\pi \in \tau$.*

In Proposition 1, we show that a neutral choice theory that nests rational choice is 1-*f*regular if and only if this choice theory is category wise rational. It follows that rational choice theory is the only neutral choice theory that nests itself, and is 1-*u*regular. In the rest of this section, we introduce category wise rationality and present the result.

Definition. *A* ***categorization*** $\mathcal{P} = \{\Omega_i\}_{i \in I}$ ***of*** Ω *is a collection of families of choice sets such that*

(1) $\bigcup_{i \in I} \Omega_i = \Omega$,

(2) *for each $i \in I$, Ω_i is closed under union, i.e. for each $S, S' \in \Omega_i$, $S \cup S' \in \Omega_i$.*

Definition. *For a given categorization $\mathcal{P} = \{\Omega_i\}_{i \in I}$, a choice function c is a* ***category wise rational with respect to*** *$\mathcal{P}$ if there is a set of rationales $\{\succeq_i\}_{i \in I}$ such that for each $S \in \Omega$ and $i \in I$, if $S \in \Omega_i$, then $c(S) = max(S, \succeq_i)$. A choice theory τ is* ***category wise rational*** *if there is a categorization $\mathcal{P}$ of Ω such that a choice function $c \in \tau$ if and only if c is category wise rational with respect to $\mathcal{P}$.*[12]

Note that category wise rationality spans a rich class of choice theories. For example, if we let the categorization $\mathcal{P}$ consist of a single category that contains all the choice sets, then we obtain the rational choice theory. On the other hand, if we let the categorization $\mathcal{P}$ consist of the singleton choice sets, then each choice function is category wise rational with respect to $\mathcal{P}$. As another example, consider a decision maker who has two rationales $\succeq_1$ and $\succeq_2$ and a guiding alternative a. From each choice set S, the decision maker chooses the $\succeq_1$-maximal alternative if $a \in S$, and the $\succeq_2$-maximal alternative if $a \notin S$. Let $\Omega_1(a) = \{S \in \Omega : a \in S\}$ and $\Omega_2(a) = \{S \in \Omega : a \notin S\}$, this procedure is category wise rational with respect to $\mathcal{P} = \{\Omega_1(a), \Omega_2(a)\}$.

Category wise rationality can be thought as a restricted version of *rationalizability by multiple rationales* (Kalai et al. (2002)). A choice function is rationalizable by multiple rationales if there is a collection of preference relations $\{\succeq_i\}_{i \in I}$ such that for each choice set S, the choice from S is $\succeq_i$-maximal for some $i \in I$. For a given category $\mathcal{P} = \{\Omega_i\}_{i \in I}$, category wise rationality additionally requires for each choice set that belongs to a particular category Ω_i, the same preference relation $\succeq_i$ be maximized. Note also that the primitive of a category wise rational choice theory is a categorization $\mathcal{P}$, whereas for given $\mathcal{P} = \{\Omega_i\}_{i \in I}$, the primitive of a category wise rational choice function is a collection of preferences $\{\succeq_i\}_{i \in I}$. Put differently, a given

[12] It is worth to note that according to the definition of a category, a choice set S might appear under different categories. If a choice function is category wise rational, then the same alternative should maximize the rationale associated with each category that contains S.

categorization $\mathcal{P}$ specifies the theory; an additionally given collection of preferences $\{\succeq_i\}_{i\in I}$ specifies a choice function that is category wise rational with respect to $\mathcal{P}$.

Proposition 1. *Let τ be a neutral choice theory that nests rational choice theory,*

i. τ is first order free regular if and only if τ is category wise rational.

ii. τ is first order universally regular if and only if τ is the rational choice theory.

Proof. See Section 7.1. □

Remark 1. A choice theory can be category wise rational with respect to several categorizations. Specifically, given a categorization, one can generate another *finer* categorization such that each category of the latter is contained in the categories of the former one. For a given neutral, 1-*f*regular choice theory that nests rational choice, the categorizations that we construct in the proof of Proposition 1 are the *coarsest* ones, that is, there is no other categorization, consistent with the given choice theory, such that each category of the former is contained in the categories of the latter.

Apesteguia & Ballester (2010) and Demuynck (2010) analyze the computational complexity of choice functions that are rationalizable by multiple rationales. They show that, in general, the verification of these theories is NP-complete. That is, the empirical refutation these models might be extremely difficult.[13] This seems in contrast with our Proposition 1. However, as we argued in the introduction, we propose choice regularities as a descriptive measure rather than as a computational one. The contrast in the results emanates from different objectives underlying the proposed measures.

[13]Since, at present, the fastest algorithm to solve any NP-complete problem has exponential worst time complexity.

4 Second order regularities and relative identification

Since choice regularities provide a plausible syntactic classification of choice axioms, we are endowed with a natural structure to constrain the choice axioms for simplicity or ease of communication. Thereof, we introduce relative identification that is useful for the identification of simple behavioral differences that distinguish a choice theory from the others.

To discipline the complexity a choice axiom, a natural candidate is first order regularities. However, in the light of Proposition 1, this seems unpromising since several boundedly rational choice theories that accommodate well-documented menu dependencies, such as attraction effect, compromise effect, or recency effect, do not satisfy any first order free regularity. Therefore, we adopt the notion of second order regularities –being inspired by the prehistoric man who counts as: "One, two and many".[14] As an example of a second order universal regularity, consider the *weak WARP* condition (Manzini & Mariotti (2007)).

wWARP: For each $a, b \in A$ and for each pair of nested choice sets $(S_1, S_2) \in \mathcal{N}$, if a is chosen from S_2 when b is available, and b is chosen from S_1 when a is available, then b should be chosen when compared to a. That is,

$$\forall a\ \forall (S_1, S_2)\ [a = c(S_1 \cup \{b\}) \wedge b = c(S_2 \cup \{a\}) \Rightarrow b = c(a, b)].$$

Since wWARP characterizes Rationalization (Cherepanov et al. (2013)) and Categorize then Choose (Manzini & Mariotti (2012)), both theories are 2-*u*regular. Rational Shortlist Method (Manzini & Mariotti (2007)) is also 2-*u*regular, since wWARP conjoint with the *expansion axiom*, which is also representable as second order regularities, characterize this choice theory. However, as it will follow from Proposition 2, there are several plausible choice theories that fail to be 2-regular. This motivates the idea of relative identification instead of full identification. Next, we introduce

[14] For example see Dixon & Blake (1983)

the formal definition of relative identification via second order regularities (free or universal). First, recall that a choice theory τ *satisfies a set of regularities* Q if each $c \in \tau$ satisfies each regularity $q \in Q$.

Definition. *Let τ_i and τ_j be two distinct choice theories, a set of 2-regularities Q_{ij}* ***identifies*** *τ_i* ***relative to*** *τ_j if τ_i satisfies Q_{ij}, but τ_j fails to satisfy Q_{ij}. Let $\mathcal{F} = \{\tau_1, \tau_2, \ldots, \tau_k\}$ be a family of choice theories, a collection of 2-regularities Q* ***relatively identifies*** *$\mathcal{F}$ if for each distinct $i, j \in \{1, \ldots, k\}$, there exists $Q_{ij} \subset Q$ that identifies τ_i relative to τ_j.*

4.1 A relative identification exercise

To motivate our relative identification exercise, consider a decision maker and an outside observer with a family of choice theories in mind that may be consistent with the observed choices of the decision maker. The observer's problem is to identify the theories that are consistent with the choices of the decision maker. If for each pair of the theories, there is a set of simple axioms that one of the theories satisfies but the other does not, then instead of going through the characterization axioms of each theory, the observer can eliminate some of the theories by falsifying these simpler axioms. We consider a specific family of boundedly rational choice theories for our relative identification exercise. Next, we briefly describe each theory in this family.

Rationalizability by Game Trees (RGT) (Xu & Zhou (2007)) The primitive of this choice procedure is an extensive form game G. Each player has a preference relation $\succ_i$ over the outcomes of the game.[15] Each alternative appears as an end node of the associated game tree only once. For each choice set S, consider the reduced game $G|S$ derived from G by retaining the paths that only lead to the terminal nodes having outcomes in S. The decision maker chooses from each choice set S, the *subgame perfect Nash equilibrium* outcome of the game $G|S$.

[15] From a decision theoretic perspective each player can be interpreted as a different self of the same decision maker, concentrating on the different aspects of the alternatives.

Choice with Limited Attention (CLA) (Masatlioglu et al. (2012)) This choice procedure has two primitives an attention filter Γ and a welfare preference $\succ$. From each choice set S, a decision maker first commits to the alternatives in $\Gamma(S)$ and then chooses the $\succ$-best alternative among these, where Γ is such that for each choice set S and $z \notin \Gamma(S)$, $\Gamma(S \setminus z) = \Gamma(S)$.[16]

Sequential rationalizability by binary rationales (SRC) (Manzini & Mariotti (2007), Apesteguia & Ballester (2013)): The primitive of this procedure is a set of binary rationales such that each rationale compares only a pair of alternatives. The decision maker removes inferior alternatives by sequentially applying this set of binary rationales according to a fixed order.

List-Rationalizable Choice (LRC) (Yildiz (2016)): LRC procedure has two primitives; an ordering of the alternatives, namely a *list*, and a complete and asymmetric *binary relation* used to compare pairs of alternatives. A *list rational* decision maker chooses from a choice set as follows. First, he orders the alternatives according to the list. Then, by using the binary relation, he compares the first and second alternatives in the list and records the winner to be compared to the next alternative. This process of carrying the current winner to the next round continues until the last alternative in the list is compared to the winner from the previous round. Winner of the last round is the alternative chosen from the entire set. A list rational decision maker uses the same non-observable list and the same binary relation to make a choice from each choice set.

4.2 Axioms and the result

Next, we introduce a set of axioms each of which is a universal second order regularity. We believe that these axioms are not only simple to verify, but also might be normatively appealing. From computational point of view, which is not our main

[16] Salant & Rubinstein (2008) present a choice procedure that has similar features in Section 5.

concern in here, only WPI and NDC are pertaining to choice sets with more than three alternatives; other axioms use only triplets and binary sets.

Weak Path Independence (WPI): For each $a, b \in A$ and $S \in \Omega$, if a is chosen from S and b is chosen from S after a is removed from S, then a is chosen when compared to b. That is,

$$\forall a \; \forall b \; \forall S \; [a = c(S) \wedge b = c(S \setminus \{a\}) \Rightarrow a = c(a, b)].$$

No Dominated Choice (NDC): For each $a, b, c \in A$ and $S \in \Omega$, if a is chosen between a, b, c and b is chosen from S when a is also available, then a is chosen when compared to b. That is,

$$\forall a \; \forall b \; \forall c \; \forall S \; [a = c(a, b, c) \wedge b = c(S \cup \{a\}) \Rightarrow a = c(a, c)].$$

Independence of the Competing Alternatives (ICA): For each $a, b \in A$ and nested choice sets $(S_1, S_2) \in \mathcal{N}$, if a is chosen from S_1 when b is available and b is chosen from S_2 when a is available, then a is also chosen from S_1 when b is removed. That is,

$$\forall a \; \forall b \; \forall (S_1, S_2) \; [a = c(S_1 \cup \{b\}) \wedge b = c(S_2 \cup \{a\}) \Rightarrow a = c(S_1 \setminus \{b\})].$$

Binary Expansion (BE): For each $a, b, c \in A$, if a is chosen in a binary comparison with b and c, then a is chosen from the triplet $\{a, b, c\}$. That is,

$$\forall a \; \forall b \; \forall c \; [a = c(a, b) \wedge a = c(a, c) \Rightarrow a = c(a, b, c)].$$

Path Existence (PE): For each $a, b, c \in A$, if a is chosen from the triplet $\{a, b, c\}$ and b is chosen when compared to a, then c is chosen when compared b. That is,

$$\forall a \; \forall b \; \forall c \; [a = c(a, b, c) \wedge b = c(a, b) \Rightarrow c = c(b, c)].$$

In Proposition 2, we focus on the aforementioned choice theories and find out all the second order free regularities that each theory satisfies. We observe that these free regularities can be represented in the form of one of the above universal regularities. It follows that this family of choice theories can be relatively identified via second order regularities that are both free and universal.

Proposition 2. *Let $\mathcal{F}$ consist of the following boundedly rational choice theories: Rationalization via Game Trees, Revealed Attention, Sequentially Rational Choice, and List Rational Choice. Let Q consist of all the second order free regularities in the form of WPI, NDC, ICA, BE or PE.*

i. For each $\tau \in \mathcal{F}$, if q is a second order free regularity that τ satisfies, then $q \in Q$.

ii. Following table shows the second order universal regularities that each theory satisfies; it follows that Q relatively identifies $\mathcal{F}$.

	WPI	*NDC*	*ICA*	*BE or PE*
RGT	✓	✓	×	✓
SRC	×	✓	×	✓
CLA	×	×	×	×
LRC	✓	✓	✓	✓

Proof. See Section 7.2. □

One can check that for each choice theory in our family, there is a choice function that does not belong to the theory although it satisfies all the second order regularities that the theory satisfies. Therefore, none of these choice theories is 2-regular. Since this is the exhaustive list of the second order regularities that these theories might satisfy, any other behavioral difference among these theories can only be represented as a third or a higher order of regularity, which we find rather difficult to interpret.

In line with the motivation for relative identification, an outsider who observes the choices of a decision maker can use the second order regularities to eliminate the theories that can not be consistent with the observed choice behavior. For example, a choice function that satisfies all the axioms but weak path independence can not be consistent with SRC, but it can be consistent with RGT. Similarly, a choice function that satisfies all the axioms but the independence of the competing alternatives, can not be consistent with RGT, but it can be consistent with LRC. Put differently, for the comparison between SRC and RGT, if we restrict ourselves to second order regular axioms, entire behavioral difference is captured by weak path independence; for a similar comparison between RGT and LRC, entire behavioral difference is captured by the independence of the competing alternatives. If we compare SRC and LRC, we observe that SRC fails to satisfy weak path independence and the independence of the competing alternatives, although LRC satisfies both. Among the theories in this family, CLA is the most permissive one in terms of the second order regularities it satisfies, since there is no second order regularity that CLA satisfies.

5 Relation to the literature

To our knowledge, this paper is the first attempt to investigate boundedly rational choice theories based on the structure of the choice axioms that these theories satisfy. On the other hand, several studies in the literature analyze the *computational complexity* of decision-making. For an incomplete list, in an early paper, Futia (1977) analyzes different heuristics in terms of computational resources such as time and attention, and argues that heuristics perform better than rational choice. More recently, Johnson (2006) uses a semiautomaton model to investigate the computational complexity of choice theories, and argues that the rational choice theory is not favored by these considerations. In the context of choice from the lists,[17] Salant (2011) considers automata implementing choice procedures, and measures complexity of a given choice procedure by the minimal number of states required for implementing

[17] In this context, the decision maker can choose different alternatives from the same choice set depending on the order according to which he encounters the set of alternatives.

it. Salant (2011) shows that *satisficing procedures* (Simon (1955)) are simpler than the rational choice procedures.[18] Mandler (2015) measures the computational burden of a choice theory by the minimum number of criteria needed to recover the choice behavior when applied sequentially. He shows that rational choice theory can be recovered by using the minimum number of criteria.

As for the difference between our study and the bulk of research based on computational complexity, mainly, we propose choice regularities to analyze the syntactic structure of a choice axiom, which is a fundamentally different question. To see this better, note that the computational complexity of a choice theory is a unique number or class that is independent of the different axiomatizations of the choice theory. Therefore, while computational complexity is silent in classifying choice axioms, choice regularities allows for such classifications.

A branch of *computational complexity theory* and of *finite model theory*, namely, *descriptive complexity theory*, which connects logic and complexity, also seems relevant to our study. Descriptive complexity theory investigates the relationship between the computational complexity of a certain problem and the logical language necessary to characterize this problem.[19] It turns out that, in some cases, it is possible to characterize complexity classes (such as *NP*) in terms of the logical languages, where there is no notion of machine, computation, or time.[20] Similarly, one can question if a choice theory being k^{th}-order regular has any implications for its computational complexity. There seems to be no direct answer following from the results in descriptional complexity, in that, these results associate the complexity classes with the languages, whereas a choice regularity is a formula in the specific language that we

[18] Our analysis is in the classical revealed preference framework in which agents choose from the choice sets. To highlight the contrast, if we map the satisficing procedures to the revealed preference framework, by augmenting the list that a decision maker follows as a primitive of the choice procedure, then the choice functions that are representable as a satisficing procedure are precisely the rational choice functions.

[19] See Immerman (2012) for an excellent survey.

[20] For example Fagin's Theorem (Fagin (1993)).

introduced in Section 2.

In another related study, Chambers et al. (2014) use tools from propositional calculus and model theory to analyze the so called *empirical content* of a choice theory, which is defined to be the theory that nests the given theory such that any data set that falsifies the former theory also falsifies the latter. They show that the empirical content of a choice theory is described by certain kinds of axioms, and provide a syntactic characterization of these axioms. Here, we propose a classification of the choice theories depending on the syntax of the choice axioms that characterize a choice theory.

The notion of choice regularities is most similar to the notion of *d-implications* introduced by Glazer & Rubinstein (2014) in the context of principal-agent problems. They study a model in which the agent is boundedly rational in his ability to understand the principal's decision rule. It is assumed that an agent of rank d can detect certain types of regularities, formulated in the form of d-implications. An agent of rank d can recognize d-implications, propositions of the form $\psi_1 \to \psi_2$ such that the antecedent ψ_1 is a conjunction of at most d atomic statements, each of which is an answer to a binary question. Glazer & Rubinstein (2014) show that for each agent of rank d, there is a sufficiently complex questionnaire that almost eliminates the probability that the agent will succeed in cheating.

6 Concluding remarks

In this study, we aim to to demonstrate that structure of a choice axiom can be analyzed fruitfully in formal terms, which we hope to open up interesting research directions for axiomatic studies. We propose choice regularities, from among other plausible formalizations, to analyze the structure of a choice axiom. We think of the contribution in two parts. First, we characterize boundedly rational choice theories rendering axiomatic characterizations that are as simple as WARP. Second, we propose and illustrate a method to identify the simple behavioral axioms that distinguish a choice theory in a given family of choice theories. As we demonstrate in the second part of the paper, obtained choice axioms that account for these behavioral differences might be difficult to infer from the axiomatic characterizations directly. Therefore, a relative identification exercise nicely complements the axiomatic characterizations of the choice theories. We believe that similar relative identification exercises for different choice theories would contribute to our understanding of the behavioral differences among different boundedly rational choice theories. We hope that researchers would find choice regularities useful in identification of the simple behavioral properties of a choice theory or to check the robustness of their axiomatic results.

7 Proofs

7.1 Proof of Proposition 1

Before proceeding to the proof of Proposition 1, we make two useful observations. First, let us start with a notational convention. Let q be a 1-$f$reg, which requires: If $a = c(S_1)$, then $b = c(S_2)$ for some $S_1, S_2 \in \Omega$, $a \in S_1$ and $b \in S_2$. If $S_2 \subset S_1$ and $a = b$, then q is said to be in the form of WARP, and denoted by $q(S_1, S_2, a)$.

Lemma 1. *Let τ be any choice theory that nests rational choice. If τ satisfies a 1-freg q, then q must be in the form of WARP.*

Proof. Let τ be any choice theory that nests rational choice. Consider any 1-$f$reg q that τ satisfies. Now, q requires: If $a = c(S_1)$, then $b = c(S_2)$ for some $a, b \in A$ and $S_1, S_2 \in \Omega$.

Step 1: We show that one must have $S_2 \subset S_1$. Suppose not, i.e. $S_2 \setminus S_1 \neq \emptyset$.

Case 1: Suppose $a = b$. Next, we show that there is a rational choice function that fails to satisfy q. Let $d \in S_2 \setminus S_1$, and consider a choice function c rationalized by a preference relation such that d is first-ranked and a is second-ranked. Note that $c(S_1) = a$, but $c(S_2) = d$.

Case 2: Suppose $a \neq b$, then consider a choice function c rationalized by a preference relation such that a is first ranked and b is bottom ranked. Note that $c(S_1) = a$, but $c(S_2) \neq b$ since there is at least one other alternative in S_2 that is preferred to b. It follows that we must have $S_2 \subset S_1$ at q.

Step 2: Now, we must have $a = b$, otherwise any rational choice function that chooses a from S_1 would not satisfy q. □

Lemma 2. *Let τ be a neutral choice theory. If τ satisfies a 1-freg $q(S_1, S_2, a)$ for some $S_1, S_2 \in \Omega$ such that $S_2 \subset S_1$ and $a \in S_2$, then for each $x \in S_2$, τ satisfies $q(S_1, S_2, x)$.*

Proof. Let $S_1, S_2 \in \Omega$ such that $S_2 \subset S_1$ and $a \in S_2$. Let τ be a neutral choice theory that satisfies $q(S_1, S_2, a)$. By contradiction, suppose that there exists $x \in S_2 \setminus \{a\}$ such that τ fails to satisfy $q(S_1, S_2, x)$. That means there exists $c \in \tau$ such that $c(S_1) = x$, but $c(S_2) \neq x$. Now, consider $\pi : A \to A$ such that $\pi(a) = x$, $\pi(x) = a$ and for each $y \in A \setminus \{x, a\}$, $\pi(y) = y$. Since τ is a neutral choice theory and $c \in \tau$, we have $c^\pi \in \tau$. However, $a = c^\pi(S_1)$, but $a \neq c^\pi(S_2)$ contradicting that τ satisfies $q(S_1, S_2, a)$. □

Now we are ready to proceed to the proof of our Proposition 1. It directly follows from Lemma 1 and the definition of being universally regular that item ii holds. In what follows we prove item i.

(*If part:*) Let τ be a category wise rational choice theory, where $\mathcal{P} = \{\Omega_i\}_{i \in I}$. Let Q be the collection of all such 1-*f*regs $q(S_1, S_2, a)$ such that there exists $i \in I$, $S_1, S_2 \in \Omega_i$ with $S_2 \subset S_1$, and $a \in S_1$. Let τ^Q be the set of choice functions identified by Q. Now we argue that a choice function $c \in \tau$ if and only if $c \in \tau^Q$. To see that $\tau \subset \tau^Q$, we show that each $c \in \tau$ satisfies each $q \in Q$. Let $c \in \tau$ and $q \in Q$, first note that q is in the form of $q(S_1, S_2, a)$, where there exists $i \in I$ with $S_1, S_2 \in \Omega_i$. Since

c is category wise rational, there exists a preference relation $\succeq_i$ such that for each $S \in \Omega_i$, $c(S) = max(S, \succeq_i)$. It directly follows that c satisfies q.

To see that $\tau^Q \subset \tau$, let c be a choice function that satisfies each $q \in Q$. Now, for each $i \in I$, consider Ω_i. Next, we construct a preference relation $\succeq_i$ such that for each $S \in \Omega_i$, $c(S) = max(S, \succeq_i)$. Let X_1 be the maximal set in Ω_i with respect to set-containment, since Ω_i is closed under union, X_1 uniquely exists. Let $x_1 = c(S_1)$. Next consider the maximal set $X_2 \in \Omega_i$ such that $x_1 \notin X_2$. Since Ω_i is closed under union, X_2 uniquely exists. Let $x_2 = c(X_2)$, and by proceeding similarly define $\{x_1 \dots x_m\}$, where $X_m = \{x_m\}$. Now, for each $x_j, x_k \in X_1$, $x_j \succeq_i x_k$ if and only if $j \leq k$. Now, for each $S \in X_i$, let $x_j = max(S, \succeq_i)$ and consider the set X_j. Since $S \subset X_j$ and c satisfies the $q(X_j, S, x_j)$, which requires if $x_j = c(X_j)$ then $x_j = c(S)$, we conclude that $x_j = c(S)$. Thus, we obtain $\tau^Q \subset \tau$.

(*Only if part:*) Let τ be a neutral 1-*f*regular choice theory that nests rational choice. Consider any 1-*f*reg q that τ satisfies, it follows from our Lemma 1 that q must be in the form of WARP, i.e. q requires: if $a = c(S_1)$, then $a = c(S_2)$ for some $S_1, S_2 \in \Omega$ with $S_2 \subset S_1$, and $a \in S_1$.

Let Q be the set of all 1-*f*regs that τ satisfies. Next, define the binary relation $\triangleright$ on Ω such that for each $S, S' \in \Omega$, $S \triangleright S'$ if and only if there exists $x \in S$ such that $q(S, S', x) \in Q$. Note that for each $S, S' \in \Omega$ to have $S \triangleright S'$ we must have $S' \subset S$. It follows that R is acyclic since set containment is an acyclic binary relation. To see that $\triangleright$ is transitive, consider $S_1, S_2, S_3 \in \Omega$ such that $S_1 \triangleright S_2$ and $S_2 \triangleright S_3$. Since $S_2 \triangleright S_3$, there exists $a \in S_3$ such that $q(S_2, S_3, a) \in Q$. Since $S_1 \triangleright S_2$, there exists $b \in S_2$ such that $q(S_1, S_2, b) \in Q$. Now, since τ is neutral, it follows from Lemma 2 that $q(S_1, S_2, a) \in Q$. Since $q(S_1, S_2, a), q(S_2, S_3, a) \in Q$, we have $q(S_1, S_3, a) \in Q$. It follows that $S_1 \triangleright S_3$. Since $\triangleright$ is a transitive binary relation on Ω, it follows from Dilworth's Theorem (Dilworth (1950)) that we can decompose Ω into a set of chains $\{\Omega_i\}_{i \in I}$ such that for each $i \in I$ and $S, S' \in \Omega_i$, we have either $S \triangleright S'$ or $S' \triangleright S$. Now, note that by construction we have $\cup_{i \in I} \Omega_i = \Omega$. Since for each $i \in I$, Ω_i is $\triangleright$-complete,

Ω_i is closed under union. It follows that $\mathcal{P}^* = \{\Omega_i\}_{i\in I}$ forms a categorization for Ω.

Next, we show that each $c \in \tau$ is category wise rational with respect to $\mathcal{P}^*$. Let $c \in \tau$ and for each $i \in I$, let $\Omega_i = \{X_{i1}, \dots X_{im}\}$ such that $X_{i1} \triangleright X_{i2} \cdots \triangleright X_{im}$, given that Ω_i contains at least two choice sets. Next, suppose that $x_{i1} = c(X_{i1})$, where X_{i1} is the maximal set in Ω_i. Let $\succeq^{x_{i1}}$ be a preference relation that top ranks x_{i1}. Next, we argue that for each choice set $X \in \Omega_i$, $c(X) = x_{i1}$. To see this first note that $X_{i1} \triangleright X_{i2}$ means that for some $x \in X_{i1}$, $q(X_{i1}, X_{i2}, x) \in Q$. Since τ is neutral, it follows from Lemma 2 that $q(X_{i1}, X_{i2}, x_{i1}) \in Q$. Therefore, we have $x_{i1} = c(X_{i2})$. By proceeding similarly we obtain that for each $X \in \Omega_i$, $c(X) = max(X, \succeq^{x_{i1}}) = x_{i1}$. It follows that c is category wise rational with respect to $\mathcal{P}^*$.

7.2 Proof of Proposition 2

It follows from Yildiz (2016) that list-rational choice theory is nested by all the other three theories we consider. Note that if a *2-freg* is satisfied by any of these theories, then it must also be satisfied by list-rational choice theory which is nested by all the others. Therefore, first we find out all the 2-*f*regs satisfied by list-rational choice theory, denoted by τ^{LRC}. Consider any 2-*f*reg q satisfied by τ^{LRC}: If $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some $a, b, c \in A$, and distinct $S_1, S_2, S_3 \in \Omega$.

Lemma 3. *If $a = b$, then q is in the form of BE, i.e. we have $S_1 = \{a, b\}, S_2 = \{a, c\}, S_3 = \{a, b, c\}$ for some $c \in A \setminus \{a, b\}$.*

Proof. First we show it is impossible to have $a = b \neq c$. Suppose not, then consider the following list rational, c: $c < .... < a$. Since for each $x \in S_1 \cup S_2 \cup S_3$, we have $x < a$, we obtain $a = c(S_1) = c(S_2)$. But since S_3 contains an alternative other than c, that one eliminates c, so we have $c \neq c(S_3)$.

Next, we show that it is possible to have $a = b = c$, only if q is in the form of BE. We first show that $S_3 \subset S_1 \cup S_2$. Suppose there is $x \in S_3 \setminus (S_1 \cup S_2)$, then consider the list rational $c : ...a < x$. For any choice of $S_1, S_2, S_3 \in \Omega$, we have $a = c(S_1) = c(S_2)$, but $c \neq c(S_3)$.

Second, we show that $S_3 \setminus S_1 \neq \emptyset$ and $S_3 \setminus S_2 \neq \emptyset$. Suppose $S_3 \subset S_1 \cap S_2$. Let $x \in S_1 \setminus S_3$, $y \in S_2 \setminus S_3$, and $z \in S_3 \setminus \{a\}$. Since $S_1 \neq S_2$, we can assume that $x \neq y$. Now, consider the list rational $c : ...z < y < x < a$, and also $z > a,\ a > y,\ x > z$. Note that in S_1 or S_2, z can eliminate a. However, x eliminates z in S_1, y eliminates z in S_2. Hence, we have $a = c(S_1) = c(S_2)$. But, since $x, y \notin S_3$, z is compared to a in the final round and eliminates a. So, we have $a \neq c(S_3)$.

Now, we know that there is $c \in S_3 \setminus S_1$ and $b \in S_3 \setminus S_2$, such that $b \neq c$, $b \in S_1$, $c \in S_2$. Finally we show that $S_1 \cup S_2 \subset \{a, b, c\}$. Suppose there is $z \in S_1 \setminus \{a, b, c\}$. Next, consider the list rational $c : ...\ z < c < b < a$, and also $z > b,\ a > z,\ a > c$. Since $c \notin S_1$ and $z > b$, z is compared to a in the final round and we have $a = c(S_1)$. Since $b \notin S_2$ and c eliminates z, c is compared to a in the final round and we have $a = c(S_2)$. But, since $b, c \in S_3$, c eliminates z, b eliminates c and then eliminates a in the final round. Hence, we have $b = c(S_3)$.

Since $S_3 \subset S_1 \cup S_2 \subset \{a, b, c\}$ and $a, b, c \in S_3$, we have $S_3 = \{a, b, c\}$, $S_1 = \{a, b\}$ and $S_2 = \{a, c\}$. It follows that q is in the form of BE. □

For the next three lemmas consider any 2-*f*reg q' satisfied by τ^{LRC} which requires: if $a = c(S_1)$ and $b = c(S_2)$, then $a = c(S_3)$ for some distinct $a, b \in A$ and distinct $S_1, S_2, S_3 \in \Omega$.

Lemma 4. *We have* $b \in S_1$.

Proof. Suppose $b \notin S_1$. As in the previous case first suppose there is $x \in S_3 \setminus S_1$, and let us pretend as if $x \neq a$ and consider the $c : ...b > a < x$ and $b > x$. Note that irrespective of $x = b$ or not, we have $a = c(S_1)$ and $b = c(S_2)$, but since x eliminates a in S_3, $a \neq c(S_3)$. So, there is $y \in S_1 \setminus S_3$. Now, let $z \in S_3 \setminus \{a\}$. If $z = b$, then consider $c : ...y < a < b$, and also $b > y$. If $z \neq b$, then consider $c : ...z < y < a < b$, and also

$z > a,\ b > \{y, z\}$. In both cases we have $a = c(S_1)$ and $b = c(S_2)$, but since b and z respectively eliminates a in S_3, $a \neq c(S_3)$. □

Lemma 5. *We have* $S_3 \subset S_1$.

Proof. Suppose not, i.e. there exists $y \in S_3 \setminus S_1$. Since $b \in S_1$, $y \neq b$. First suppose $S_1 \setminus S_2 \neq \emptyset$, and let $x \in S_1 \setminus S_2$. If $x \neq a$, then consider the list rational c: ... $b < x < a < y$, and $b > \{a, y\}$. Since $x \in S_1$, x eliminates b, and compared to a in the final round, so we have $a = c(S_1)$. Since $x \notin S_2$, and b eliminates a and y, we have $b = c(S_2)$. But since y eliminates a, we have $a \neq c(S_3)$.

If $x = a$, then consider the list rational c: ... $b < a < y$, and $b > y$. Since $y \notin S_1$, we have $a = c(S_1)$. Since $a \notin S_2$, and b eliminates y, we have $b = c(S_2)$. But since y eliminates a, we have $a \neq c(S_3)$.

Next, suppose $S_2 \setminus S_1 \neq \emptyset$, and let $z \in S_2 \setminus S_1$. Now, consider the list rational c: ... $a < z < b > y$, and also $a > b,\ y > a$. Since $z, y \notin S_1$, we have $a = c(S_1)$. Since z eliminates a in S_2, and b eliminates z and y, we have $b = c(S_2)$. But since y eliminates a, we have $a \neq c(S_3)$. □

Lemma 6. *If q' is not in the form of NDC or ICA, then we have* $S_2 \subset S_1$.

Proof. Suppose q' is not in the form of NDC or ICA but there exists $x \in S_2 \setminus S_1$. By Lemma 4, we have $b \in S_1$, so $x \neq b$. Since, by Lemma 3, $S_3 \subset S_1$ and $S_1 \neq S_3$, there is $y \in S_1 \setminus S_3$. Since $x \notin S_1$, we have $x \neq y$.

A. Suppose there is $c \in S_3$ with $c \notin \{a, b\}$. Since $S_3 \subset S_1$ and $x \notin S_1$, we have $c \neq x$. Since $y \notin S_3$, $c \neq y$ either. Now, we are left with two possibilities. If $y \neq b$, then a, b, x, y, c are all distinct. If we must have $y = b$, then it follows that $S_1 = S_3 \cup \{b\}$.

A1. Suppose $y \neq b$, so a, b, x, y, c are all distinct. Now, consider the list rational c: ...$c < y < a < x < b$, also $c > a,\ a > b$, and $\{x, b\} > \{c, y\}$. Since $y \in S_1$, y eliminates c. For $c(S_1)$, since $x \notin S_1$, a eliminates b, and we have $a = c(S_1)$. For $c(S_2)$, note that

x is compared to b in the final round, so we have $b = c(S_2)$. For $c(S_3)$, since $y \notin S_3$, c always eliminates a.

A2. Now, suppose $S_1 = S_3 \cup \{b\}$. First, suppose that $S_3 = \{a, c\}$. Then $S_1 = \{a, b, c\}$. Since q' is not in the form of NDC, we have $a \notin S_2$. Now, consider the list rational c: $...c < b < a$, and also $c > a$. We have $a = c(S_1)$ and $b = c(S_2)$, but since c eliminates a, $a \neq c(S_3)$.

Second, suppose that there is $z \in S_3 \setminus \{a, c\}$. Since $S_3 \subset S_1$ and $x \notin S_1$, $z \neq x$. Since $b \notin S_3$, $z \neq b$. So, we know that a, b, c, x, z are all distinct. Moreover, we have $S_3 = S_1 \setminus \{b\}$. Since q' is not in the form of ICA, there must be $w \in S_1 \setminus S_2$. Now, there are two possibilities $w = a$ or $w \neq a$. First suppose $w = a$ and consider c^l : $...c < x < b < a$, also $c > \{a, w\}$, and $\{x, b\} > c$. For $c(S_1)$, b eliminates c, a eliminates b, and we get $a = c(S_1)$. For $c(S_2)$, x eliminates c, b eliminates x. Since $a \notin S_2$, we have $b = c(S_2)$. For $c(S_3)$, since $x, b \notin S_3$ and c eliminates a, $a \neq c(S_3)$.

Next, suppose $w \neq a$. Now we might have $w \in S_3$, but either $w \neq c$ or $w \neq z$. Assume w.l.o.g. that $w \neq c$ and consider c^l: $...c < x < b < w < a$, also $c > \{a, w\}$ $b > a$, and $\{x, b\} > c$. For $c(S_1)$, b eliminates c, w eliminates b, and we get $a = c(S_1)$. For $c(S_2)$, x eliminates c, b eliminates x and a. Since $w \notin S_2$, we have $b = c(S_2)$. For $c(S_3)$, since $x, b \notin S_3$ and c eliminates w and a, $a \neq c(S_3)$.

B. Suppose $S_3 = \{a, b\}$. Since we assume that $S_3 = \{a, b\}$ and $x \notin S_1$, we have $y \notin \{a, b, x\}$. So, we know that a, b, x, y are all distinct. Now, consider the list rational c: $...y < x < b > a$, and also $y > b,\ a > y$. For $c(S_1)$, y eliminates b and compared to a in the final round. Since a eliminates y, we have $a = c(S_1)$. For $c(S_2)$, since x eliminates y and b eliminates x and a, we have $b = c(S_2)$. But for $c(S_3)$, since b eliminates a, $a \neq c(S_3)$. $\square$

Now, let us consider any 2-$f$reg satisfied by τ^{LRC}. We focus on two cases separately First we consider the case where at least one of S_1, S_2, S_3 contains more than three alternatives.

Lemma 7. *For each 2-freg q satisfied by τ^{LRC}, if at least one of S_1, S_2, S_3 contains more than three alternatives, then q is in the form of WPI, NDC or ICA.*

Proof. Consider any 2-freg q that requires: if $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some $a, b, c \in A$ and distinct $S_1, S_2, S_3 \in \Omega$ where at least one of them contains more three alternatives. Since at least one of the sets contains more three alternatives, q is not BE. It follows from Lemma 3 that $a \neq b$. In the rest of the proof we show that if q is not in the form of NDC or ICA, then q must be in the form of WPI.

Step 1: We show that $c \in \{a, b\}$. Suppose this is not true. Since $S_1 \neq S_2$, assume w.l.o.g. that $S_1 \setminus S_2 \neq \emptyset$. Now, there are two possibilities:

A. Suppose there is $x \in S_1 \setminus S_2$ such that $x \neq c$. Now, there are two possibilities: $x = a$ or $x \neq a$. If $x = a$, then consider c^l : $...c < b < a$. If $x \neq a$, then consider c^l : $...c < b < x < a$, and also $a > x$, $b > a$. Note that in both cases we have $a = c(S_1)$ and $b = c(S_2) = b$, but $c \neq c(S_3)$.

B. Suppose $S_1 \setminus S_2 = \{c\}$. Since for some $i \in \{1, 2, 3\}$, S_i has more than three alternatives, there exists $d \in S_i$ such that $d \notin \{a, b, c\}$.

B1. Suppose $d \in S_3$. Then, consider c^l : $...c > b > d < a$, also $a > c$, and $b > a$. Regardless of which other choice set might also contain d, and whether $b \in S_1$ or not: For $c(S_1)$, first c eliminates b, then a eliminates c and d, so we have $a = c(S_1)$. For $c(S_2)$, since $c \notin S_2$, b eliminates d and a, so we have $b = c(S_2)$. For $c(S_3)$, since d eliminates c, $c \neq c(S_3)$.

B2. Suppose there is no such $d \in S_3$, but $d \in S_2$. It follows that $S_3 \subset \{a, b, c\}$

i. If $d \notin S_1$, then consider c^l : $...a < d < b$, and $a > b$. For $c(S_1)$, since $d \notin S_1$ and a eliminates b, we have $a = c(S_1)$. For $c(S_2)$, first d eliminates a, then b eliminates d and we have $b = c(S_2)$. For $c(S_3)$, since $S_3 \subset \{a, b, c\}$ and both a and b eliminate c, $c \neq c(S_3)$.

ii. If $d \in S_1$, then consider c^l: $...d < c < a > b$ also $d > a$, $b > c$ and $b > d$ For

$c(S_1)$, since $c \in S_1$, first c eliminates d, then a eliminates c and possibly b, so we have $a = c(S_1)$. For $c(S_2)$, since $c \notin S_2$ and $d \in S_2$, first d eliminates a, then b eliminates d, so we have $b = c(S_2)$. For $c(S_3)$, since $S_3 \subset \{a, b, c\}$ and both a and b eliminate c, $c \neq c(S_3)$.

B3. Suppose there is no such $d \in S_3 \cup S_2$, but $d \in S_1$. It It follows that $S_2 \cup S_3 \subset \{a, b, c\}$ Now consider c^l : ...$b < d < a$, and $b > a$. For $c(S_1)$, since $d \in S_1$, first d eliminates b, then a eliminates d, and we get $c(S_1) = a$. For $c(S_2)$, since $d \notin S_2$, b eliminates a, and we get $c(S_2) = b$. For $c(S_3)$, since $S_3 \subset \{a, b, c\}$ and both a and b eliminate c, $c \neq c(S_3)$.

Finally, we can conclude that $c \in \{a, b\}$. For the rest assume w.l.o.g. that $c = a$, where $a = c(S_1)$.

Step 2: Since q is not in the form of NDC or ICA, it follows from Lemma 5 and Lemma 6 that $S_2 \cup S_3 \subset S_1$.

Step 3: We show that $S_3 = \{a, b\}$. Suppose not, i.e. there is $c \in S_3 \setminus \{a, b\}$. By the previous step, we know that $S_2 \subset S_1$ and $S_3 \subset S_1$.

A. Suppose there is $x \in S_1 \setminus (S_2 \cup S_3)$, so $x \neq c$. Since $b = c(S_2)$ and $a = c(S_3)$, $x \notin \{a, b\}$ either. It follows that a, b, c, x are all distinct. Now, consider c^l : ...$b < x > c > a$, also $b > \{c, a\}$ and $a > x$. For $c(S_1)$, since $x \in S_1$, first x eliminates b and c, then a eliminates x, and we get $a = c(S_1)$. For $c(S_2)$, since $x \notin S_2$, b eliminates a and c, and we get $b = c(S_2)$. But, for $c(S_3)$ since $x \notin S_2$ and both b and c eliminate a, we get $a \neq c(S_3)$.

B. Suppose $S_1 = S_2 \cup S_3$. Let $x \in S_1 \setminus S_2$ and $y \in S_1 \setminus S_3$. We know that $x \neq y$, $x \in S_3$ and $y \in S_2$. It follows that $x \neq b$ and $y \notin \{a, c\}$. Moreover since S_1 must have at least four elements we can not both have $x = a$ and $y = b$.

i. Suppose that $x = a$ but $y \neq b$, and consider c^l : ...$c < y > b < a$ and $c > a$. For $c(S_1)$, first y eliminates c and b, then a eliminates y and we get $a = c(S_1)$. For $c(S_2)$, since $y \in S_2$ and $a \notin S_2$, first y eliminates the rest, then b eliminates y and we get $b = c(S_2)$. But, for $c(S_3)$ since c eliminates b and a, we get $a \neq c(S_3)$.

ii. Suppose that $x \neq a$ but $y = b$, and consider c^l : ...$c < b < x < a$, also $c > \{a, x\}$, and $b > a$ For $c(S_1)$, we clearly have $a = c(S_1)$. For $c(S_2)$, since $x \notin S_2$ and b eliminates c and a, and we have $b = c(S_2)$. But, For $c(S_3)$ since $b \notin S_3$ and c eliminates both x and a, we have $a \neq c(S_3)$.

iii. Finally suppose both $x \neq a$ and $y \neq b$, and consider c^l : ...$c < y > b < x < a$, also $c > a,\ c > x,\ b > a$, and $c > b$. For $c(S_1)$, note that x is the alternative compared to a in the last round. Since a eliminates x, we have $a = c(S_1)$. For $c(S_2)$, first y eliminates c and b eliminates y, then since $x \notin S_2$, b eliminates a and we get $b = c(S_2)$. But, for $c(S_3)$ since $y \notin S_3$ and c eliminates a, b and x, we have $a \neq c(S_3)$.

Step 4: We show that $S_2 = S_1 \setminus \{a\}$. We already know that $S_2 \subset S_1$. Suppose there is $x \in S_1 \setminus S_2$ such that $x \neq a$. Consider c^l : ...$b < x < a$ and $b > a$. We clearly have $a = c(S_1)$, $b = c(S_2)$. For $c(S_3)$, since by Step 3 $S_3 = \{a, b\}$, b eliminates a and we get $a \neq c(S_3)$. It follows that $S_1 = S_2 \cup \{a\}$. Since $S_1 \neq S_2$, we have $S_2 = S_1 \setminus \{a\}$. □

Lemma 8. *For each 2-freg q satisfied by τ^{LRC}, if S_1, S_2 and S_3 contains at most three alternatives, then q must be in the form of WPI , NDC, PE or BE.*

Proof. First suppose that $a = b$. It follows from Lemma 3 that q is in the form of BE. Next, let us consider any 2-$f$reg, q where $a \neq b$. Now, suppose $c \in \{a, b\}$. Assume w.l.o.g. that $c = a$. By Lemma 4, we have $S_2 \cup S_3 \subset S_1$. Since S_1 contains at most three alternatives, we have $S_1 = \{a, b, c\}$ for some $c \in A \setminus \{a, b\}$. Moreover, since S_1, S_2, S_3 are distinct, there are two possibilities, either $S_2 = \{a, b\}$ and $S_3 = \{a, c\}$ or $S_2 = \{b, c\}$ and $S_3 = \{a, b\}$. For the first possibility, we obtain a 2-$f$reg in the form of NDC, for the second we obtain a 2-$f$reg in the form of PE.

Next, suppose $c \notin \{a, b\}$. We show that q must be in the form of PE. We first show that $a \in S_2$. Suppose not, then consider the list rational c: $c < ... < b < a,\ a > c$. Clearly we have $a = c(S_1)$. Since, $a \notin S_2$, we have $b = c(S_2)$. But since each alternative can eliminate c, $c \neq c(S_3)$. Next, we show that $b \in S_1$. Suppose not, then consider the choice function above with $b > a$. Similarly, we obtain the desired contradiction.

Next, we show that either $S_1 \subset S_2$ or $S_2 \subset S_1$. Suppose not, and let $x \in S_1 \setminus S_3$ and $y \in S_2 \setminus S_1$. Since each set has at most three alternatives, we have $S_1 = \{a, b, x\}$ and $S_2 = \{a, b, y\}$. If $c \neq x$, then consider the list rational $c : c < ... < b < x < a,\ b > a$. If $c = x$, then consider the list rational $c : c < ... < a < y < b,\ a > b$. By similar reasoning, we have $a = c(S_1)$ and $b = c(S_2)$, but $c \neq c(S_3)$.

Assume w.l.o.g. that $S_2 \subset S_1$. It follows that there is $x \in S_1 \setminus \{a, b\}$. Next, we show that $x = c$. Suppose not, then consider the list rational $c : c < ... < b < x < a,\ b > a$. Once more we get a contradiction. Hence, we obtain $S_1 = \{a, b, c\}$ and $S_2 = \{a, b\}$

Finally, we show that $S_3 \subset S_1$. Suppose not and let $x \in S_3 \setminus S_1$. Then consider the list rational $c : ...x < b < c < a,\ b > a,\ x > c$. Once more we get a contradiction. Hence, we obtain $S_3 \subset \{a, b, c\}$. Now, we can have $S_3 = \{a, c\}$ or $S_3 = \{b, c\}$. If the former holds then q contradicts NDC. It follows that $S_3 = \{b, c\}$ and q is in the form of PE. □

Now we are ready to proceed to the proof of Proposition 2.

i. Consider any 2-*f*reg q satisfied by τ^{LRC} that requires: If $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some $a, b, c \in A$, and distinct $S_1, S_2, S_3 \in \Omega$. First suppose that at least one of S_1, S_2, S_3 contains more than three alternatives, then, it follows from Lemma 6 that q is in the form of WPI, NDC or ICA. Next suppose each of S_1, S_2 and S_3 contains at most three alternatives, then it follows from Lemma 8 that q is in the form of WPI, NDC, PE or BE.

ii. We leave this part to the reader since it is straightforward to verify that each theory satisfies or violates the corresponding axioms indicated in the table. Since for each pair of theories, there is at least one axiom that is satisfied by one of the theories and not by the other, we obtain relative identification via second order regularities for this family of choice theories.

References

Apesteguia, J. & Ballester, M. A. (2010), 'The computational complexity of rationalizing behavior', *Journal of Mathematical Economics* **46**(3), 356–363. 4, 13

Apesteguia, J. & Ballester, M. A. (2013), 'Choice by sequential procedures', *Games and Economic Behavior* **77**(1), 90–99. 7, 16

Chambers, C. P., Echenique, F. & Shmaya, E. (2014), 'The axiomatic structure of empirical content', *The American Economic Review* **104**(8), 2303–2319. 21

Cherepanov, V., Feddersen, T. & Sandroni, A. (2013), 'Rationalization', *Theoretical Economics* **8**(3), 775–800. 14

Cohen, H. & Lefebvre, C. (2005), *Handbook of categorization in cognitive science*, Elsevier. 7

Crossley, J. N., Ash, C. J., Brickhill, C. J. & Stillwell, J. (2012), *What is mathematical logic?*, Courier Corporation. 8

Demuynck, T. (2010), 'The computational complexity of boundedly rational choice behavior'. 4, 13

Dilworth, R. P. (1950), 'A decomposition theorem for partially ordered sets', *Annals of Mathematics* pp. 161–166. 24

Dixon, R. M. & Blake, B. J. (1983), *Handbook of Australian languages*, Vol. 3, John Benjamins Publishing. 14

Fagin, R. (1993), 'Finite-model theory-a personal perspective', *Theoretical computer science* **116**(1), 3–31. 20

Futia, C. (1977), 'The complexity of economic decision rules', *Journal of Mathematical Economics* **4**(3), 289–299. 4, 19

Glazer, J. & Rubinstein, A. (2014), 'Complex questionnaires', *Econometrica* **82**(4), 1529–1541. 21

Immerman, N. (2012), *Descriptive complexity*, Springer Science & Business Media. 20

Johnson, M. R. (2006), Economic choice semiautomata: Structure, complexities and aggregations, *in* 'Econometric Society World Congress, London'. 4, 19

Kalai, G., Rubinstein, A. & Spiegler, R. (2002), 'Rationalizing choice functions by multiple rationales', *Econometrica* **70**(6), 2481–2488. 7, 12

Luce, R. D. & Raiffa, H. (1957), 'Games and decisions: Introduction and critical survey (new york, 1957)', *Chs. vi and xiv* . 6

Mandler, M. (2015), 'Rational agents are the quickest', *Journal of Economic Theory* **155**, 206–233. 4, 20

Manzini, P. & Mariotti, M. (2007), 'Sequentially rationalizable choice', *American Economic Review* **97(5)**, 1824 1839. 7, 14, 16

Manzini, P. & Mariotti, M. (2012), 'Categorize then choose: Boundedly rational choice and welfare', *Journal of the European Economic Association* **10**(5), 1141–1165. 14

Masatlioglu, Y., Nakajima, D. & Ozbay, E. Y. (2012), 'Revealed attention', *The American Economic Review* **102**(5), 2183–2205. 7, 16

Salant, Y. (2011), 'Procedural analysis of choice rules with applications to bounded rationality', *American Economic Review* **101.2**, 724–748. 4, 19, 20

Salant, Y. & Rubinstein, A. (2008), 'Choice with frames', *Review of Economic Studies* **75**, 1287–1296. 16

Samuelson, P. (1938), 'A note on the pure theory of consumer's behaviour', *Economica* **5 (1)**, 61–71. 3

Sen, A. (1993), 'Capability and well-being73', *The quality of life* p. 30. 6

Simon, H. A. (1955), 'A behavioral model of rational choice', *The quarterly journal of economics* pp. 99–118. 20

Xu, Y. & Zhou, L. (2007), 'Rationalizability of choice functions by game trees', *Journal of Economic Theory* **134**, 548–556. 7, 15

Yildiz, K. (2016), 'List-rationalizable choice', *Theoretical Economics* **11**(2), 587–599. 7, 16, 25