A New Efficiency Criterion for Probabilistic Assignments and Implications for the Probabilistic Serial Rule

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May 10, 2017
joint with...
Motivation

We analyze assignment problems which constitute a basis for many applications, such as:

- Matching firms and workers
- Matching schools and students
- Matching hospitals and interns

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- each agent’s *ex-ante* evaluation of a probabilistic assignment is his expected utility for that probabilistic assignment.
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- **ex-ante (utilitarian) efficiency**: the probabilistic assignment maximizes the sum of the expected utilities.
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▶ agents are endowed with vNM-preferences over probabilistic assignments.

▶ ex-ante (utilitarian) efficiency: the probabilistic assignment maximizes the sum of the expected utilities.

▶ in many applications, agents are asked to report their preference orderings, i.e. cardinal preferences are unobservable.
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**Part 1** We characterize this new efficiency notion.

**Part 2** When and how to improve the PS mechanism based on this efficiency notion?
A set of $n$ agents: $N = \{1, \ldots, n\}$

A set of $n$ objects: $A = \{a, b, \ldots\}$

Ordinal preferences: $\{R_i\}_{i \in N}$ weak orders on $A$
Deterministic assignments

\[ \mu : N \rightarrow A \quad \text{one-to-one} \]
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Probabilistic assignments

\[ \pi : \text{lottery over deterministic assignments}. \]
\[ \pi_i : \text{lottery that } i \text{ receives}. \]
\[ \pi_i(a) : \text{probability that } i \text{ is assigned } a. \]
Given ordinal preferences $R = \{ R_i \}_{i \in N}$ on $A$, $u = \{ u_i \}_{i \in N}$ is consistent with $R$ if

$$\forall i \in N \quad \forall a, b \in A \quad a R_i b \iff u_i(a) \geq u_i(b)$$
Social Welfare Efficiency
Given ordinal preferences $R$, when should we say that $\pi$ dominates $\pi'$ in efficiency terms?
sd-efficiency

$$\pi \text{ sd-dominates } \pi' \text{ at } R \text{ if for each pair } i \in N \text{ and } a \in A,$$

$$\sum_{b: bR_ia} \pi_i(b) \geq \sum_{b: bR_ia} \pi'_i(b)$$

and for at least one pair, the inequality is strict.
**sd-efficiency**

\[ \pi \text{ sd-dominates } \pi' \text{ at } R \text{ if for each pair } i \in N \text{ and } a \in A, \]

\[ \sum_{b: bR_i a} \pi_i(b) \geq \sum_{b: bR_i a} \pi'_i(b) \]

and for at least one pair, the inequality is strict.

\[ \pi \in \Pi \text{ is sd-efficient if it is not sd-dominated.} \]
The **ex-ante utilitarian social welfare** at \((u, \pi)\):

\[
SW(u, \pi) = \sum_{i \in N} E[u_i(\pi_i)] = \sum_{(i,a) \in N \times A} \pi_i(a) u_i(a).
\]

If \(\pi \in \text{argmax}_{\pi \in \Pi} SW(u, \pi)\), then \(\pi\) is **ex-ante efficient** at \(u\).
Question

If $\pi$ is sd-efficient at $R$, then is there a $u$ consistent with $R$ such that $\pi$ is ex-ante (utilitarian) efficient at $u$?
Theorem (McLennan (2002)):

\[ \pi \text{ is sd-efficient at } R \quad \text{iff} \quad \exists \ u \text{ consistent with } R \text{ s.t.} \]
\[ \pi \in \arg\max_{\pi \in \Pi} SW(u, \pi) \]
Motivation

- McLennan (2002), *Ordinal efficiency and the polyhedral separating hyperplane theorem*,

- Manea (2008), *A constructive proof of the ordinal efficiency welfare theorem*,

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A new efficiency notion

\[ \mathcal{U}(R, \pi) \]: set of utility profiles consistent with \( R \) and at which \( \pi \) is ex-ante efficient.
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$\pi$ weakly sw-dominates $\pi'$ $\iff$ $\mathcal{U}(R, \pi') \subseteq \mathcal{U}(R, \pi)$
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Thm (McLennan, 2002):

\( \pi \) is sd-efficient at \( R \) \iff \( \mathcal{U}(R, \pi) \neq \emptyset \)
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\[ \pi \text{ is sd-efficient at } R \iff \mathcal{U}(R, \pi) \neq \emptyset \]

Corollary: Each \( \pi \) that is sd-efficient sw-dominates each \( \pi' \) that is not sd-efficient.
characterization of sw-domination

Support of $\pi$: $Sp(\pi) = \{(i, a) \in N \times A : \pi_i(a) > 0\}$
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**Proposition:** Let $\pi, \pi' \in \Pi$ and $R \in \mathcal{R}$. The assignment $\pi$ sw-dominates $\pi'$ at $R$ if and only if

1. $\pi' \notin P^{sd}(R)$ and $\pi \in P^{sd}(R)$, or
2. $\pi' \in P^{sd}(R)$ and $Sp(\pi, R) \subset S_p(\pi', R)$. 
Observation:

i. PS assignment does not always **sd-dominate** the RP assignment.

ii. PS assignment always **weakly sw-dominates** the RP assignment.
For each \((i, a), (j, b) \in N \times A\), \((i, a) \sim_{(\pi, R)} (j, b)\) if \(\pi_i(b) > 0\) and \(a \preceq_i b\).
Extended Support

- For each \((i, a), (j, b) \in N \times A\), \((i, a) \sim_{(\pi, R)} (j, b)\) if \(\pi_i(b) > 0\) and \(a I_i b\).

- A cycle at \(\sim_{(\pi, R)}\):
  \[
  (i_1, a_1) \sim_{(\pi, R)} (i_2, a_2) \sim_{(\pi, R)} \cdots \sim_{(\pi, R)} (i_k, a_k) \sim_{(\pi, R)} (i_1, a_1).
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**Definition:** A pair \((i, a) \in N \times A\) is in the **extended support** of \(\pi\) relative to \(R\), denoted by \((i, a) \in ExtSp(\pi, R)\), if there is a cycle of \(\sim_{(\pi, R)}\) that contains \((i, a)\).
Proposition: For each $\pi, \pi' \in \Pi$, $\pi$ sw-dominates $\pi'$ at $R \in \mathcal{R}$ if and only if

i. $\pi' \notin P^{sd}(R)$ and $\pi \in P^{sd}(R)$, OR

ii. $\pi' \in P^{sd}(R)$ and $ExtSp(\pi) \subset ExtSp(\pi')$. 
Corollary:

1. **Strict preferences:** $\pi$ is sw-efficient $\iff$ $\pi$ is deterministic Pareto efficient.

2. **Weak preferences:** $\pi$ is sw-efficient $\iff$ $\pi$ is sd-efficient and each agent is indifferent between the objects he is assigned with positive probability.
Main take away:
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*Use minimal amount of randomization to establish fairness.*
Implications for the Probabilistic Serial Mechanism
The PS mechanism

- An assignment mechanism is a function $\varphi : \mathcal{R} \rightarrow \Pi$.
- For each $R \in \mathcal{R}^S$, the PS assignment is computed as follows:
  - Consider each object as an infinitely divisible good with a one unit supply that will be eaten by agents in the time interval $[0, 1]$ as follows:
    - Step 1: Each agent eats from his most preferred object at the same speed. When an object is completely eaten, proceed to the next step.
    - Steps $s \geq 2$: Each agent eats from his most preferred object from among the ones that have not yet been completely eaten at the same speed. When an object is completely eaten, proceed to the next step.
- The algorithm terminates when all the objects are exhausted.
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Example

Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

\[
\begin{array}{ccc|ccc}
R_1 & R_2 & R_3 & \pi^{ps}(R) & a & b & c \\
\end{array}
\]
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Let \( N = \{1, 2, 3\} \) and \( A = \{a, b, c\} \).

\[
\begin{array}{ccc|ccc}
R_1 & R_2 & R_3 & \pi^{ps}(R) & a & b & c \\
\hline
a & a & b & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4}
\end{array}
\]
Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

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Question: Given $R \in \mathcal{R}^S$, is the PS assignment sw-efficient in the class of sd-envy-free assignments at $R$?
Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

The assignment $\pi$, which is sd-envy-free at $R$, sw-dominates $PS(R)$ at $R$, since $Sp(\pi) \subsetneq Sp(\pi^{ps}(R))$. 
**Definition:** For each $R \in \mathcal{R}^S$, $G(R)$ is a directed graph where:

1. Each agent-object pair is a vertex.
2. For each vertex pair, $(i, a) \rightarrow (j, b)$, if for each pair of objects $x, y \in A$ such that $x R_i a$ with $\pi_{ps}(i, x) > 0$ and $b P_j y$ with $\pi_{ps}(j, y) > 0$, we have $x P_j y$. 
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Lemma: For each $a \in A$ and $(i, a), (j, a) \in V$, if $(i, a) \not\leftrightarrow (j, a)$, then there exists $\epsilon_{ij} > 0$ such that

$$\pi^{ps}(j, U(R_j, a)) > \pi^{ps}(i, U(R_j, a)) + \epsilon_{ij}.$$
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**Definition:** $G(R)$ is $a$-connected if for each $i, j \in N$ such that $\pi_{ps}(R)(i, a) > 0$, $(i, a)$ is connected to $(j, a)$ in $G(R)$. The graph $G(R)$ is connected if it is $a$-connected for each $a \in A$. 
Example 1

For each \( x \in A \), \( G(R) \) on \( N \times \{x\} \) is a complete graph.
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For each $x \in A$, $G(R)$ on $N \times \{x\}$ is a complete graph.

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For each $x \in A$, $G(R)$ on $N \times \{x\}$ is a star-graph.

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Example 2

For each \( x \in A \), \( G(R) \) on \( N \times \{x\} \) is a star-graph.

\[
\begin{array}{ccccc}
R \\
\hline
1 & 2 & 3 & 4 & 5 \\
\hline
a & b & c & d & e \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
**Definition:** $G(R)$ is $a$-connected if for each $i, j \in N$ such that $\pi^{ps}(R)(i, a) > 0$, $(i, a)$ is connected to $(j, a)$ in $G(R)$. The graph $G(R)$ is connected if it is $a$-connected for each $a \in A$.

**Proposition:** If $G(R)$ is connected, then the $PS$ assignment is strongly sw-efficient among the sd–envy-free assignments at $R$. 
Betweenness

We introduce a property which turns out to be critical in understanding when connectedness is necessary for the $PS$ assignment to be strongly sw-efficient among the sd–envy-free assignments.
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**Definition:** A preference profile \( R \in \mathcal{R}^S \) satisfies **betweenness** if for each pair \( a, b \in A \) that are simultaneously exhausted in the PS algorithm at \( R \) and for each \( i \in N \) with \( \pi_{ps}(i, a) > 0 \), there exists \( c \in A \) such that \( \pi_{ps}(i, c) > 0 \) and \( a P_i c P_i b \).
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We introduce a property which turns out to be critical in understanding when connectedness is necessary for the PS assignment to be strongly sw-efficient among the sd–envy-free assignments.

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Examples of \( R \) that satisfies betweenness:

- If for each distinct \( a, b \in A \), \( a \) and \( b \) are exhausted at different times.
- Each agent-object pair is matched with positive probability.
Proposition: For each $R \in \mathcal{R}^S$ that satisfies betweenness, if the $PS$ assignment is strongly sw-efficient among the sd–envy-free assignments, then $G(R)$ is connected.
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The assignment \( \pi \), which is sd-envy-free at \( R \), sw-dominates \( PS(R) \) at \( R \), since \( S_p(\pi) \subsetneq S_p(\pi^{ps}(R)) \).
1. Each object is exhausted at different times in $\pi^{ps}(R)$, so betweenness is satisfied.

2. Consider the graph $G(R)$
3. $G(R)$ is $a$-connected and $c$-connected. However, $G(R)$ is not $b$-connected.

4. Since $(1, b) \not\rightarrow (3, b)$, we can transfer some amount of $b$ from 3 to 1 without violating sd–envy-freeness.
## Improving PS in sw-terms

<table>
<thead>
<tr>
<th>$R_1$</th>
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<th>$R_3$</th>
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<th>$b$</th>
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5. Transfer the assignment of $b$ from 3 to 1 until any additional transfer makes 3 to envy 1, i.e. transfer $1/4$ probability of $b$ from 3 to 1.
Improving PS in sw-terms

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6. Transfer the assignment of $b$ from 3 to 1 until any additional transfer makes 3 to envy 1, i.e. transfer $\frac{1}{4}$ probability of $b$ from 3 to 1.

7. Add the $c$ share of 1 to the assignment of 3. Thus, we obtain the assignment $\pi$, which is sd–envy-free and sw-dominates the PS assignment.
conclusion
conclusion
Thank you!