A New Efficiency Criterion for Probabilistic Assignments and Implications for the Probabilistic Serial Rule

Battal Doğan [1] Serhat Doğan [2] Kemal Yıldız [3]

[1]University of Lausanne

[2]Bilkent University

[3]Bilkent University

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joint with...

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- agents are endowed with vNM-preferences over probabilistic assignments.
- ex-ante (utilitarian) efficiency: the probabilistic assignment maximizes the sum of the expected utilities.
- in many applications, agents are asked to report their preference orderings, i.e. cardinal preferences are unobservable.

outline

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Preliminaries

A set of n agents: $N = \{1, \ldots, n\}$

A set of n objects: $A = \{a, b, \ldots\}$

Ordinal preferences: $\{R_i\}_{i \in N}$ weak orders on A

Deterministic assignments

$$\mu : N \to A \qquad \text{one-to-one}$$

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Probabilistic assignments

π: lottery over deterministic assignments.

π_i: lottery that i receives.

$\pi_i(a)$: probability that i is assigned a.

Given ordinal preferences $R = \{R_i\}_{i \in N}$ on A, $u = \{u_i\}_{i \in N}$ is consistent with R if

$$\forall i \in N \quad \forall a, b \in A \quad a \; R_i \; b \Longleftrightarrow u_i(a) \geq u_i(b)$$

Social Welfare Efficiency

Given ordinal preferences R, when should we say that π dominates π' in efficiency terms?

sd-efficiency

π sd-dominates π' at R if for each pair $i \in N$ and $a \in A$,

$$\sum_{b:bR_ia} \pi_i(b) \geq \sum_{b:bR_ia} \pi'_i(b)$$

and for at least one pair, the inequality is strict.

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and for at least one pair, the inequality is strict.

$\pi \in \Pi$ is sd-efficient if it is not sd-dominated.

The ex-ante utilitarian social welfare at (u, π):

$$SW(u, \pi) = \sum_{i \in N} E[u_i(\pi_i)] = \sum_{(i,a) \in N \times A} \pi_i(a)\ u_i(a).$$

If $\pi \in argmax_{\pi \in \Pi} SW(u, \pi)$, then π is ex-ante efficient at u.

Question

If π is sd-efficient at R, then is there a u consistent with R such that π is ex-ante (utilitarian) efficient at u?

Theorem (McLennan (2002)):

π *is sd-efficient at* R *iff* $\exists$ *u consistent with* R *s.t.* $\pi \in argmax_{\pi \in \Pi} SW(u, \pi)$

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- Manea (2008), A constructive proof of the ordinal efficiency welfare theorem,

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$$\pi \text{ sw-dominates } \pi' \iff \mathcal{U}(R,\pi') \subsetneq \mathcal{U}(R,\pi)$$

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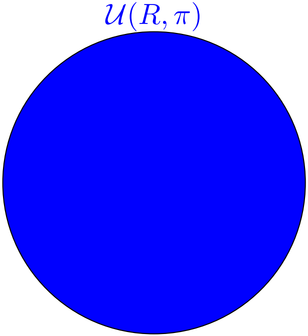

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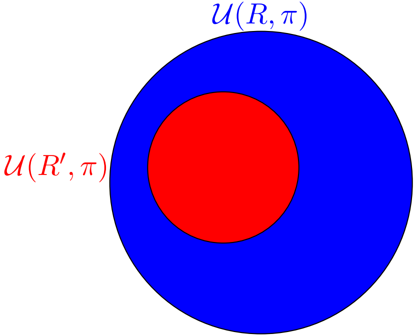

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π is strongly sw-efficient at R if it is not sw-dominated at R.

Thm (McLennan, 2002):

π *is sd-efficient at* $R \iff \mathcal{U}(R, \pi) \neq \emptyset$

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Corollary: *Each* π *that is sd-efficient sw-dominates each* π' *that is not sd-efficient.*

characterization of sw-domination

Support of π: $Sp(\pi)= \{(i, a) \in N \times A : \pi_i(a) > 0\}$

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Proposition: *Let* $\pi, \pi' \in \Pi$ *and* $R \in \mathcal{R}$*. The assignment* π *sw-dominates* π' *at* R *if and only if*

1. $\pi' \notin P^{sd}(R)$ *and* $\pi \in P^{sd}(R)$*, or*
2. $\pi' \in P^{sd}(R)$ *and* $Sp(\pi, R) \subsetneq Sp(\pi', R)$*.*

Observation:

i. *PS assignment does not always sd-dominate the RP assignment.*

ii. *PS assignment always weakly sw-dominates the RP assignment.*

Extended Support

- For each $(i, a), (j, b) \in N \times A$, $(i, a) \sim_{(\pi, R)} (j, b)$ if $\pi_i(b) > 0$ and $a \; I_i \; b$.

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- For each $(i,a),(j,b) \in N \times A$, $(i,a) \sim_{(\pi,R)} (j,b)$ if $\pi_i(b) > 0$ and $a \; I_i \; b$.
- A cycle at $\sim_{(\pi,R)}$:
  $(\mathbf{i_1}, \mathbf{a_1}) \sim_{(\pi,R)} (i_2, a_2) \sim_{(\pi,R)} \ldots \sim_{(\pi,R)} (i_k, a_k) \sim_{(\pi,R)} (\mathbf{i_1}, \mathbf{a_1})$.

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Definition: *A pair $(i,a) \in N \times A$ is in the extended support of π relative to R, denoted by $(i,a) \in ExtSp(\pi,R)$, if there is a cycle of $\sim_{(\pi,R)}$ that contains (i,a).*

Weak Preferences

Proposition: *For each $\pi, \pi' \in \Pi$, π sw-dominates π' at $R \in \mathcal{R}$ if and only if*

i. $\pi' \notin P^{sd}(R)$ *and* $\pi \in P^{sd}(R)$, **OR**

ii. $\pi' \in P^{sd}(R)$ *and* $ExtSp(\pi) \subsetneq ExtSp(\pi')$.

Corollary:

1. *Strict preferences:* π *is sw-efficient* $\iff$ π *is deterministic Pareto efficient.*
2. *Weak preferences:* π *is sw-efficient* $\iff$ π *is sd-efficient and each agent is indifferent between the objects he is assigned with positive probability.*

Main take away:

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Use minimal amount of randomization to establish fairness.

Implications for the Probabilistic Serial Mechanism

The PS mechanism

- An assignment mechanism is a function $\varphi : \mathcal{R} \to \Pi$,
- For each $R \in \mathcal{R}^S$, the *PS* assignment is computed as follows:
  - Consider each object as an infinitely divisible good with a one unit supply that will be eaten by agents in the time interval $[0, 1]$ as follows:

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Steps $s \geq 2$: Each agent eats from his most preferred object from among the ones that have not yet been completely eaten at the same speed. When an object is completely eaten, proceed to the next step.

  - The algorithm terminates when all the objects are exhausted

Example

Let $N = \{1,2,3\}$ and $A = \{a,b,c\}$.

R_1	R_2	R_3

$\pi^{ps}(R)$	a	b	c

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Definition: *An assignment* π *is* *sd–envy-free* *at* R *if for each pair of agents* $i, j \in N$, π_i *sd-dominates* π_j *at* R_i.

Definition: *An assignment π is sd–envy-free at R if for each pair of agents $i, j \in N$, π_i sd-dominates π_j at R_i.*

Question: Given $R \in \mathcal{R}^S$, is the *PS* assignment *sw-efficient* in the class of *sd-envy-free* assignments at R?

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Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

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π	a	b	c
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3	0	$\frac{1}{2}$	$\frac{1}{2}$

The assignment π, which is sd-envy-free at R, sw-dominates $PS(R)$ at R, since $Sp(\pi) \subsetneq Sp(\pi^{ps}(R))$.

Definition: *For each $R \in \mathcal{R}^S$, $G(R)$ is a directed graph where:*

1. *Each agent-object pair is a vertex.*
2. *For each vertex pair, $(i, a) \rightarrow (j, b)$, if for each pair of objects $x, y \in A$ such that $x \; R_i \; a$ with $\pi^{ps}(i, x) > 0$ and $b \; P_j \; y$ with $\pi^{ps}(j, y) > 0$, we have $x \; P_j \; y$.*

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Lemma: *For each* $a \in A$ *and* $(i, a), (j, a) \in V$, *if* $(i, a) \not\to (j, a)$, *then there exists* $\epsilon_{ij} > 0$ *such that* $\pi^{ps}(j, U(R_j, a)) > \pi^{ps}(i, U(R_j, a)) + \epsilon_{ij}$.

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Definition: *$G(R)$ is a-connected if for each $i, j \in N$ such that $\pi^{ps}(R)(i, a) > 0$, (i, a) is connected to (j, a) in $G(R)$. The graph $G(R)$ is connected if it is a-connected for each $a \in A$.*

Example 1

R

1	2	3	4
a	a	a	a
b	b	b	b
c	c	c	c
d	d	d	d

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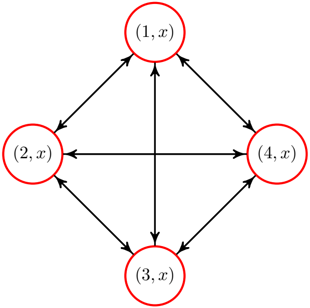
R

1	2	3	4
a	a	a	a
b	b	b	b
c	c	c	c
d	d	d	d

For each $x \in A$, $G(R)$ on $N \times \{x\}$ is a complete graph.



Example 2

R

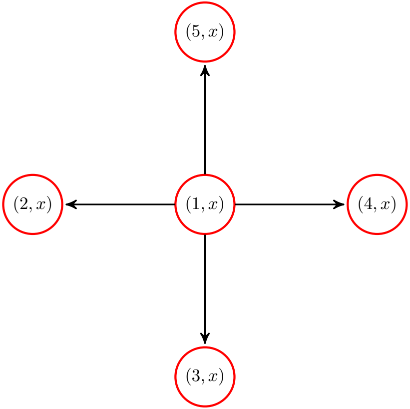
1	2	3	4	5
a	b	c	d	e
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Example 2

For each $x \in A$, $G(R)$ on $N \times \{x\}$ is a star-graph.

R

1	2	3	4	5
a	b	c	d	e
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.



Sufficiency

Definition: *$G(R)$ is a-connected if for each $i, j \in N$ such that $\pi^{ps}(R)(i, a) > 0$, (i, a) is connected to (j, a) in $G(R)$. The graph $G(R)$ is connected if it is a-connected for each $a \in A$.*

Proposition: *If $G(R)$ is connected, then the PS assignment is strongly sw-efficient among the sd–envy-free assignments at R.*

Betweenness

We introduce a property which turns out to be critical in understanding when connectedness is necessary for the *PS* assignment to be strongly sw-efficient among the sd–envy-free assignments.

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Definition: *A preference profile $R \in \mathcal{R}^S$ satisfies betweenness if for each pair $a, b \in A$ that are simultaneously exhausted in the PS algorithm at R and for each $i \in N$ with $\pi^{ps}(i, a) > 0$, there exists $c \in A$ such that $\pi^{ps}(i, c) > 0$ and $a \; P_i \; c \; P_i \; b$.*

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Examples of R that satisfies betweenness:

- If for each distinct $a, b \in A$, a and b are exhausted at different times.
- Each agent-object pair is matched with positive probability.

Necessity

Proposition: *For each $R \in \mathcal{R}^S$ that satisfies betweenness, if the PS assignment is strongly sw-efficient among the sd–envy-free assignments, then $G(R)$ is connected.*

Example

Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.

R_1	R_2	R_3
a	a	b
b	c	c
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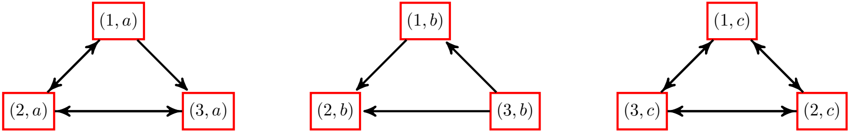
The assignment π, which is sd-envy-free at R, sw-dominates $PS(R)$ at R, since $Sp(\pi) \subsetneq Sp(\pi^{ps}(R))$.

Improving PS in sw-terms

1. Each object is exhausted at different times in $\pi^{ps}(R)$, so betweenness is satisfied.

2. Consider the graph $G(R)$



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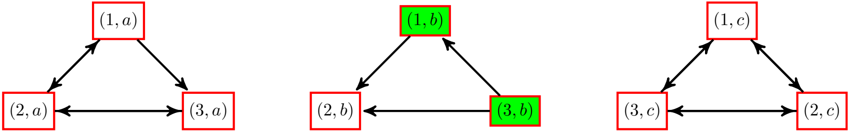

3. $G(R)$ is a-connected and c-connected. However, $G(R)$ is not b-connected.

4. Since $(1, b) \not\rightarrow (3, b)$, we can transfer some amount of b from 3 to 1 without violating sd–envy-freeness.

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R_1	R_2	R_3
a	a	b
b	c	c
c	b	a

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5. Transfer the assignment of b from 3 to 1 until any additional transfer makes 3 to envy 1, i.e. transfer $1/4$ probability of b from 3 to 1.

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6. Transfer the assignment of b from 3 to 1 until any additional transfer makes 3 to envy 1, i.e. transfer $1/4$ probability of b from 3 to 1.

7. Add the c share of 1 to the assignment of 3. Thus, we obtain the assignment π, which is sd–envy-free and sw-dominates the *PS* assignment.

conclusion

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Thank you!