Efficiency and Stability of Probabilistic Assignments in Marriage Problems*

Battal Doğan† Kemal Yıldız‡

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Abstract

We study marriage problems where two groups of agents, men and women, match each other and probabilistic assignments are possible. When only ordinal preferences are observable, stochastic dominance efficiency (sd-efficiency) is commonly used. First, we provide a characterization of sd-efficient allocations in terms of a property of an order relation defined on the set of man-woman pairs. Then, using this characterization, we constructively prove that for each probabilistic assignment that is sd-efficient for some ordinal preferences, there is a von Neumann-Morgenstern utility profile consistent with the ordinal preferences for which the assignment is Pareto efficient. Our second result pertains to stability. We show that when the preferences are strict, for each ordinal preference profile and each probabilistic assignment that is ex-post stable, there is a von Neumann-Morgenstern utility profile consistent with the ordinal preferences for which the assignment belongs to the core: No coalition can deviate to another probabilistic assignment among its members and achieve a higher total expected utility.

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†Faculty of Business and Economics, University of Lausanne; battaldogan@gmail.com.

‡Department of Economics, Bilkent University; kemal.yildiz@bilkent.edu.tr.

1 Introduction

The theory of two-sided matching problems has been useful in providing solutions to many real-life economic problems (see Roth and Sotomayor [21]). A marriage problem, which constitutes a basis for two-sided matching problems, consists of two equal-sized groups of agents: men and women. Each man has ordinal preferences over women and vice versa. A majority of the literature is focused on deterministic assignments, where men and women are matched one-to-one. One can also think of probabilistic assignments, which are lotteries over deterministic assignments, and these are interesting for at least two reasons: (1) agents may match in fractions; for example, a consultant may allocate his time among consulting for several firms,[1] (2) probabilistic assignments may help us achieve fairness when it is not possible with deterministic assignments (e.g. Bogomolnaia and Moulin [5]). Here, we consider probabilistic assignments, and in particular their "stability" and "efficiency".

We consider a model where only ordinal preference information is available. That is, for each man, all we know is his preference ordering over women, and vice versa, which is in line with the applications and the theoretical literature. Most allocation rules that have been discussed in the theoretical literature elicit only ordinal preferences, as opposed to utility information over possible mates.[2] When only ordinal preference information is available, extensively studied efficiency and stability notions are *stochastic dominance efficiency* (*sd-efficiency*) and *ex-post stability*. We ask whether probabilistic assignments that are sd-efficient or ex-post stable for ordinal preferences are possibly efficient or stable for cardinal preferences.

The main contribution of this paper pertains to stability, which is a central robustness condition for two-sided matchings, and requires that no pair of agents has an incentive to circumvent the matching. A deterministic assignment is stable if no pair of a man and a woman prefers each other to their assigned mates. A probabilistic assignment is ex-post stable if it can be expressed as a lottery over stable deterministic assignments. In the deterministic case, if an assignment is stable, then it is also in the core of the associated non-transferable utility game (NTU-game). It follows that in

[1] See Manjunath [17] and Bogomolnaia and Moulin [6] for more examples of fractional matching markets.

[2] One justification for why such mechanisms are common is that it is a complex process for an agent to formulate his utility information. See Bogomolnaia and Moulin [5] for a detailed discussion.

an ordinal environment where monetary transfers are not allowed, if an assignment is ex-post stable, then there is no incentive for ex-post[3] group deviation. However, ex-ante,[4] a group of men and women may consider to break away from the rest of society and may get matched among themselves to make each member better off in terms of expected utility. This possibility motivates us to ask whether ex-post stability implies welfare properties deducible from the ordinal preferences to avoid such break aways.

We show that when preferences are strict (no agent is indifferent between two different agents), for each ex-post stable probabilistic assignment, there is a utility profile consistent with the ordinal preferences such that no group of agents consisting of equal numbers of men and women can deviate to a probabilistic assignment among themselves and make each member better off. In fact, we prove an even stronger result (Theorem 2) by showing that for each ex-post stable probabilistic assignment, there is a utility profile consistent with the ordinal preferences such that no group of agents consisting of equal numbers of men and women can deviate to a probabilistic assignment among themselves in which the sum of their expected utilities is greater. Put differently, we associate with each utility profile a transferable utility game (TU-game), as in Shapley and Shubik [23], such that a coalition is admissible if it consists of equal numbers of men and women, and the value of each admissible coalition is the maximal total expected utility the coalition can achieve by getting matched among themselves.[5] We show that for each ex-post stable assignment, there is a utility profile such that the initial assignment belongs to the core of the associated TU-game (Theorem 2). That is, even if we were to allow monetary transfers, there would not be any profitable ex-ante group deviation.

Our other result pertains to efficiency. A natural efficiency requirement for probabilistic assignments is sd-efficiency, which is based on the first-order stochastic dominance relation. A probabilistic assignment stochastically dominates (sd-dominates) another if for each agent, the probability distribution assigned to that agent in the former first-order stochastically dominates the probability distribution assigned to that agent in the latter assignment. Assignments that are undominated in this sense are

[3]After the realization of a deterministic assignment that is assigned a positive probability by the probabilistic assignment.

[4]That is, before the realization of a particular deterministic assignment.

[5]We assume preferences to be quasilinear in money.

called "sd-efficient".[6] We ask the following question: Consider an ordinal preference profile and an sd-efficient assignment. Does there necessarily exist a utility profile consistent with the ordinal preferences at which the assignment is Pareto efficient? We show that for each ordinal preference profile, and for each sd-efficient assignment, one can construct a utility profile consistent with the ordinal preferences such that the sum of the expected utilities of the agents is maximized at that assignment (Theorem 1). To prove the result, we characterize the sd-efficiency of an assignment in terms of a property of an order relation that we define over the set of man-woman pairs.[7]

Our results regarding efficiency are intimately related to some recent results in the literature. Carroll [7] proves a counterpart of Theorem 1 in a more general social choice setup, from which Theorem 1 can be obtained as a corollary. Aziz et al. [2] provide an interesting non-expected utility generalization of Carroll [7]. Our results are based on order theoretical analysis; we characterize the sd-efficiency of an assignment in terms of the acyclicity of a binary relation on the set of man-woman pairs, which parallels the results by Bogomolnaia and Moulin [5] and by Katta and Sethuraman [12]. In this vein, Aziz et al. [2] note that it does not seem possible to extend this characterization to Carroll's [7] or their own general social choice setting. Moreover, the utility profile constructed in Theorem 1 clearly relates to the utility profile constructed in Theorem 2, which sheds further light on the relation between sd-efficiency and ex-post stability.[8]

Our results for stability, unlike efficiency, can not be directly related to the recent findings on the general social choice setup. To prove Theorem 2, we observe an interesting property of ex-post stable assignments related to the lattice structure of stable deterministic assignments (see Knuth [14], pp. 92-93, who attributes the discovery of this lattice structure to J. H. Conway). In Proposition 3, we show that each ex-post stable probabilistic assignment can be decomposed into a collection of deterministic stable assignments, which can be ordered in a way that each man's welfare is non-increasing and each woman's welfare is non-decreasing as we follow the assignments

[6] This notion is usually referred to as "ordinal efficiency", starting from Bogomolnaia and Moulin [5]. Here, we use the terminology of Thomson [25].

[7] Bogomolnaia and Moulin [5] has a similar characterization for the problem of assigning objects in the case of strict preferences; Katta and Sethuraman [12] has a similar characterization in the case of weak preferences.

[8] In fact, as noted in the proof of Proposition 2, if the ex-post stable assignment has a decomposition into stable assignments such that no agent matches with the same agent in two different assignments, then the two utility profiles coincide.

from the first to the last. This result, which we show via the *rounding approach* due to Sethuraman and Teo [24], plays a key role in the proof of Theorem 2. A corollary of Theorem 2 is that ex-post stability implies sd-efficiency when preferences are strict, which is independently shown also by Manjunath [16].

1.1 Related Literature

For the probabilistic assignment setup, a strand of the literature discusses different stability notions and their relation to each other. At the heart of this literature lies the notion of ex-post stability, which is also at the centre of our study. Roth et al. [20], Rothblum [22], and Vande Vate [26] provide a characterization of the set of ex-post stable assignments that we summarize in Lemma 2 and use in the proof of Proposition 3. More recently, Manjunath [16] investigates several ex-ante stability and core notions that are based on first order stochastic dominance. Kesten and Ünver [13] propose ex-ante stability notions for the school choice problem, which is different from the marriage problem in two ways. First, school's have *priority orderings*, rather then preferences, which are not taken into consideration in the welfare analysis. Second, schools are assumed to behave non-strategically. In all these studies, the focus is on stability and core notions based on ordinal preference information. In contrast, we are interested in the relationship between ex-post stability based on ordinal preferences and a core notion based on cardinal preferences.

Following Bogomolnaia and Moulin [5], one strand of the literature studies the probabilistic assignment of objects. In this literature, there is a result similar to our Theorem 1 for the problem of allocating objects, due to McLennan [18].[9] We show that McLennan's result follows from our Theorem 1 (Corollary 1). Manea [15] gives a constructive proof of McLennan's result. Our proof technique is similar to Manea's: An order relation, the acyclicity of which characterizes an assignment's sd-efficiency, is used to construct the utility profile. As shown in Manea [15], if only one side of the market has preferences over the other, then the relation used to characterize sd-efficiency rather directly delivers the desired utility profile. However, in marriage problems, the preferences of both men and women must be taken into account, both in sd-efficiency and utilitarian social welfare considerations. We show that doing so requires a rather

[9] Athanassoglou [1] approaches the problem in McLennan [18] by using duality.

novel and deliberate construction of the utility profile compared to the case of one-sided markets.

In the deterministic assignment literature, the theory of stable assignments without transfers (NTU-model) was first developed by Gale and Shapley [11]. Given preference rankings, the algorithm proposed by Gale and Shapley [11] selects a stable assignment that is Pareto optimal for the proposing side among all stable assignments. On the other hand, the theory of stable assignments with transfers (TU-model) was developed by Shapley and Shubik [23], where for given valuations of agents, the core of the associated TU-game (the assignment game) is characterized.[10]

Our Theorem 2 offers a connection between TU-models where monetary transfers are allowed, and NTU-models, where such transfers are not allowed. Echenique [8] and Echenique et al. [9] offer a similar connection between TU-models and NTU-models based on the observable content of stability in both setups. Their analysis is in a *revealed preference framework*, in which ordinal preferences are not observable and are recovered from an observed *aggregate matching*.[11] Echenique et al. [9] argue that, in this setting, the matching theory with transfers is nested in the matching theory without transfers, that is, for each aggregate matching, if there is a utility profile such that the matching is in the core of the associated TU-game, then there is an ordinal preference profile such that the matching is the unique stable matching. For our Theorem 2, we assume that ordinal preferences and an ex-post stable assignment are observable, but utility profiles are not observable. We construct a utility profile that is consistent with the ordinal preferences such that the given ex-post stable matching is in the core of the associated TU-game. Since the converse statement does not hold,[12] as compared with Echenique [8], in our setting we observe that the matching theory without transfers is nested in the matching theory with transfers.

Another recent study, which provides several interesting insights on the relationship between "stability in a TU-model" and "stability in an NTU-model", is Echenique and Galichon [10]. One goal of their study is to understand for which stable deterministic assignments, availability of monetary transfers would not affect the stability. Among several other interesting results, they shows that, for a particular subset of stable de-

[10]This theory is applied to the marriage problem by Becker [3].
[11]See Echenique et al. [9] for the definition of an aggregate matching
[12]See our Example 2.

terministic assignments, which they call the set of isolated assignments, for any ordinal preference profile, one can construct a utility profile such that each isolated stable deterministic assignment is stable even when monetary transfers are allowed. At the end of Section 3.3, we show that their result follows from our Theorem 2.

2 The Framework

Let M be a set of n men and W a set of n women. Each $i \in M$ has preferences over W, and each $j \in W$ has preferences over M. Let $N = M \cup W$. For each $i \in N$, the preferences of i, which we denote by R_i, is a **weak order**, i.e. R_i is transitive and complete. Let P_i denote the associated strict preference relation, and I_i the associated indifference relation. Let $\mathcal{R}_i$ denote the set of all possible preference relations for i, and $\mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$ denote the set of all possible preference profiles.

A **deterministic assignment** is a one-to-one function $\mu : M \cup W \to M \cup W$ such that for each $(m, w) \in M \times W$, we have $\mu(m) \in W$, $\mu(w) \in M$, and $\mu(m) = w$ if and only if $\mu(w) = m$. A deterministic assignment can be represented by an $n \times n$ matrix, with rows indexed by men and columns indexed by women, and having entries in $\{0, 1\}$, such that each row and each column has exactly one 1. Such a matrix is called a **permutation matrix**. For each $(m, w) \in M \times W$, having 1 in the (m, w) entry indicates that m is assigned to w. A **probabilistic assignment** is a probability distribution over deterministic assignments. A probabilistic assignment can be represented by an $n \times n$ matrix having entries in $[0, 1]$ such that the sum of the entries in each row and each column is 1. Such a matrix is a **doubly stochastic matrix**. For each probabilistic assignment π, and each pair $(m, w) \in M \times W$, the entry π_{mw} indicates the probability that m is assigned to w at π. Since each doubly stochastic matrix can be represented as a convex combination of permutation matrices (Birkhoff [4] and von Neumann [27]), the set of all doubly stochastic matrices provides another representation for the set of all probabilistic assignments. Let Π be the set of all doubly stochastic matrices.

We denote the collection of all lotteries over M by $\mathcal{L}(M)$, and the collection of all lotteries over W by $\mathcal{L}(W)$. For each $i \in M$ with preferences $R_i \in \mathcal{R}_i$, a von-Neumann–Morgenstern (vNM) utility function u_i is a real-valued mapping on W, i.e. $\boldsymbol{u_i : W \to \mathbb{R}}$. We obtain the corresponding preferences of i over $\mathcal{L}(W)$ by comparing

expected utilities. For each $i \in M$ with preferences $R_i \in \mathcal{R}_i$, a vNM utility function u_i **is consistent with $\boldsymbol{R_i}$** if for each pair $(w, w') \in W$ we have $u_i(w) \geq u_i(w')$ if and only if $w \; R_i \; w'$. For each woman, a (vNM) utility function consistent with her ordinal preferences is defined similarly.

For each utility profile $u = (u_i(.))_{i \in N}$ and probabilistic assignment π, the **utilitarian social welfare at $(\boldsymbol{u}, \boldsymbol{\pi})$** is the sum of the expected utilities of the agents, that is:

$$SW(u, \pi) = \sum_{(m,w) \in M \times W} \pi_{mw}(u_m(w) + u_w(m))$$

An assignment **$\boldsymbol{\pi}$ is ex-ante utilitarian-welfare maximizing at a utility profile $\boldsymbol{u}$** if it maximizes the social welfare at u, i.e. $\pi \in \text{argmax}_{\pi \in \Pi} SW(u, \pi)$.

Next, we define a well-known notion of efficiency that is independent of any vNM utility specification consistent with the ordinal preferences. Let $\pi, \pi' \in \Pi$ and $R \in \mathcal{R}$; we say that **$\boldsymbol{\pi}$ first-order stochastically dominates $\boldsymbol{\pi'}$ at $\boldsymbol{R}$** if for each pair $(m, w) \in M \times W$:

$$\sum_{w' : w' R_m w} \pi_{mw'} \geq \sum_{w' : w' R_m w} \pi'_{mw'} \text{ and } \sum_{m' : m' R_w m} \pi_{m'w} \geq \sum_{m' : m' R_w m} \pi'_{m'w}$$

such that for at least one pair, at least one of the inequalities is strict. An assignment **$\boldsymbol{\pi} \in \boldsymbol{\Pi}$ is sd-efficient at $\boldsymbol{R}$** if no probabilistic assignment sd-dominates π at R. For each $R \in \mathcal{R}$, let $P^{sd}(R)$ denote the set of sd-efficient assignments at R.

3 Results

3.1 A characterization of sd-efficiency

For each pair $(\pi, R) \in \Pi \times \mathcal{R}$, we define two relations $\sim_{(\pi,R)}$ and $\succ_{(\pi,R)}$ on $M \times W$ induced by (π, R), and characterize the sd-efficiency of an assignment π at R in terms of a property of $\sim_{(\pi,R)}$ and $\succ_{(\pi,R)}$.

For each pair $(m, w), (m', w') \in M \times W$, $(m, w) \sim_{(\pi,R)} (m', w')$ if and only if $\pi_{mw} > 0$, $\pi_{m'w'} > 0$, and $w' \; I_m \; w$, $m \; I_{w'} \; m'$. For each pair $(m, w), (m', w') \in M \times W$, $(m, w) \succ_{(\pi,R)} (m', w')$ if and only if $\pi_{m,w} > 0$, $\pi_{m'w'} > 0$, and $w' \; R_m \; w$, $m \; R_{w'} \; m'$ with at least one strict preference. Note that if assignment π is deterministic and preferences

are strict, then $(m,w) \succ_{(\pi,R)} (m',w')$ implies that m and w' prefer each other to their assigned mates. Furthermore, suppose that $(m,w) \succ_{(\pi,R)} (m',w') \succ_{(\pi,R)} (m,w)$. Then, m and m' (or w and w') can be better off by exchanging mates. In general, if the relation $\succ_{(\pi,R)}$ has a cycle, agents in the cycle can Pareto improve by exchanging mates along this cycle. Next, we formulate an acyclicity requirement on $\succeq_{(\pi,R)}$ with the same implication for sd-efficiency.

Note that if $(m,w) \succ_{(\pi,R)} (m',w')$, then (m,w) and (m',w') are not related according to $\sim_{(\pi,R)}$, i.e. $(m,w) \nsim_{(\pi,R)} (m',w')$. Let the relation $\succeq_{(\pi,R)}$ be the union of the two relations we have just defined. That is, $\succeq_{(\pi,R)} = \sim_{(\pi,R)} \cup \succ_{(\pi,R)}$. A **strong cycle of $\succeq_{(\pi,R)}$** is a sequence of pairs $(m_1,w_1),(m_2,w_2), \ldots,(m_k,w_k) \in M \times W$ such that $(m_1,w_1) \succ_{(\pi,R)} (m_2,w_2) \succeq_{(\pi,R)} \ldots \succeq_{(\pi,R)} (m_k,w_k) \succeq_{(\pi,R)} (m_1,w_1)$. The relation $\succeq_{(\pi,R)}$ **is weakly acyclic** if and only if it has no strong cycle. Next, we characterize sd-efficient assignments. This result generalizes the characterization of sd-efficiency of object assignments on the strict preference domain due to Bogomolnaia and Moulin [5] and on the weak preference domain due to Katta and Sethuraman [12].

Proposition 1. *An assignment $\pi \in \Pi$ is sd-efficient at $R \in \mathcal{R}$ if and only if $\succeq_{(\pi,R)}$ is weakly acyclic.*

Proof. Let $\pi \in \Pi$, $R \in \mathcal{R}$.

Only-if part: We prove the contrapositive statement. Suppose that $\succeq_{(\pi,R)}$ is not weakly acyclic, that is, there is a sequence of pairs $(m_1,w_1),(m_2,w_2), \ldots,(m_k,w_k) \in M \times W$ such that $(m_1,w_1) \succ_{(\pi,R)} (m_2,w_2) \succeq_{(\pi,R)} \ldots \succeq_{(\pi,R)} (m_k,w_k) \succeq_{(\pi,R)} (m_1,w_1)$. Let $\epsilon \equiv \min_{i\in\{1,\ldots,k\}} \pi_{m_i w_i}$. Let $\pi' \in \Pi$ be defined by setting for each $i \in \{1,\ldots,k\}$, $\pi'_{m_i w_i} = \pi_{m_i w_i} - \epsilon$, $\pi'_{m_i w_{i+1}} = \pi_{m_i w_{i+1}} + \epsilon$ (with the convention that $w_{k+1} = w_1$), and for each other pair (m,w), $\pi'_{mw} = \pi_{mw}$. Note that π' sd-dominates π at R. Thus, $\pi \in P^{sd}(R)$.

If part: We prove the contrapositive statement. Suppose that $\pi \notin P^{sd}(R)$, that is, there is $\pi' \in \Pi$, which sd-dominates it. Without loss of generality, suppose that there is a man, say $m_1 \in M$, who is better off at π' in stochastic dominance terms. Then, there are $w_1, w_2 \in W$ such that $w_2 \; P_{m_1} \; w_1$, $\pi'_{m_1 w_2} > \pi_{m_1 w_2}$, and $\pi'_{m_1 w_1} < \pi_{m_1 w_1}$. Moreover, there is $m_2 \in M$ such that $m_1 \; R_{w_2} \; m_2$ and $\pi'_{m_2 w_2} < \pi_{m_2 w_2}$. Note that $(m_1,w_1) \succ_{(\pi,R)} (m_2,w_2)$. Now, there are $m_3 \in M$, $w_3 \in W$ such that $w_3 \; R_{m_2} \; w_2$, $\pi'_{m_2 w_3} > \pi_{m_2 w_3}$, $m_2 \; R_{w_3} \; m_3$, and $\pi'_{m_3 w_3} < \pi_{m_3 w_3}$. Then, $(m_1,w_1) \succ_{(\pi,R)} (m_2,w_2) \succeq_{(\pi,R)}$

(m_3, w_3). Proceeding inductively, we can add pairs to this sequence, and since there are finitely many man-woman pairs, this sequence should include a cycle of $\succeq_{(\pi,R)}$.

Note that if the cycle includes (m_1, w_1), then it is strong, and if the cycle is strong, we are done. So, suppose that it is not a strong cycle. Let the cycle consist of $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in M \times W$. Note that $(x_1, y_1) \sim_{(\pi,R)} (x_2, y_2) \sim_{(\pi,R)} \ldots \sim_{(\pi,R)} (x_k, y_k) \sim_{(\pi,R)} (x_1, y_1)$. Remember that we also have $\pi'_{x_1y_1} < \pi_{x_1y_1}, \pi'_{x_2y_2} < \pi_{x_2y_2}, \ldots, \pi'_{x_ky_k} < \pi_{x_ky_k}$, and $\pi'_{x_1y_2} > \pi_{x_1y_2}, \pi'_{x_2y_3} > \pi_{x_2y_3}, \ldots, \pi'_{x_ky_1} > \pi_{x_ky_1}$. Let $\epsilon_1 \equiv \min\{\pi_{x_1y_1} - \pi'_{x_1y_1}, \ldots, \pi_{x_ky_k} - \pi'_{x_ky_k}\}$, $\epsilon_2 \equiv \min\{\pi'_{x_1y_2} - \pi_{x_1y_2}, \ldots, \pi'_{x_ky_1} - \pi_{x_ky_1}\}$. Let $\epsilon \equiv \min\{\epsilon_1, \epsilon_2\}$. Let $\pi'' \in \Pi$ be defined by setting for each $i \in \{1, \ldots, k\}$, $\pi''_{x_iy_i} = \pi'_{x_iy_i} + \epsilon$, $\pi''_{x_iy_{i+1}} = \pi'_{x_iy_{i+1}} - \epsilon$ (with the convention that $y_{k+1} = y_1$), and for each other pair (m, w), $\pi''_{mw} = \pi'_{mw}$. Since for each $i \in \{1, \ldots, k\}$, $y_i \; I_{x_i} \; y_{i+1}$ and $x_{i-1} \; I_{y_i} \; x_i$, π'' also sd-dominates π. Also, by the construction of π'', there are two consecutive pairs in the cycle, say (x_t, y_t) and (x_{t+1}, y_{t+1}), such that either $\pi''_{x_ty_t} = \pi_{x_ty_t}$ or $\pi''_{x_ty_{t+1}} = \pi_{x_ty_{t+1}}$.

Now, because π'' sd-dominates π, as we did above, we can find a cycle of $\succeq_{(\pi,R)}$, say (x'_1, y'_1), (x'_2, y'_2), $\ldots, (x'_t, y'_t) \in M \times W$, such that for each $i \in \{1, \ldots, t\}$, $\pi''_{x'_iy'_i} < \pi_{x'_iy'_i}$ and $\pi'_{x'_iy'_{i+1}} > \pi_{x'_iy'_{i+1}}$ (with the convention that $y_{t+1} = y_1$). Note that (x_t, y_t) and (x_{t+1}, y_{t+1}) can not be part of this cycle consecutively. Continuing similarly, we will obtain additional assignments that sd-dominate π and additional cycles. None of those cycles can include (x_t, y_t) and (x_{t+1}, y_{t+1}) consecutively; and for each additional cycle, we will identify additional consecutive pairs in the cycle that can not be part of any cycle in the future. But since the number of cycles in $\succeq_{(\pi,R)}$ is finite, continuing this process eventually leads to a strong cycle. Thus, $\succeq_{(\pi,R)}$ is not weakly acyclic.

□

Note that, given $(\pi, R) \in \Pi \times \mathcal{R}$, for each $\pi \in P^{sd}(R)$, the relation $\succeq_{(\pi,R)}$ can have cycles of the form $(m_1, w_1) \succeq_{(\pi,R)} (m_2, w_2) \succeq_{(\pi,R)} \ldots \succeq_{(\pi,R)} (m_k, w_k) \succeq_{(\pi,R)} (m_1, w_1)$. However, $\succeq_{(\pi,R)}$ being weakly acyclic implies that such a cycle should belong to $\sim_{(\pi,R)}$, i.e. the cycle should be of the form $(m_1, w_1) \sim_{(\pi,R)} (m_2, w_2) \sim_{(\pi,R)} \ldots \sim_{(\pi,R)} (m_k, w_k) \sim_{(\pi,R)} (m_1, w_1)$.

Given $(\pi, R) \in \Pi \times \mathcal{R}$, let $C_{\pi,R}$ be the binary relation on $M \times W$, defined as follows: For each pair $(m, w), (m', w') \in M \times W$, $(m, w) \; C_{\pi,R} \; (m', w')$ if and only if there is a cycle of $\sim_{(\pi,R)}$ that contains both, i.e. there is a sequence of pairs (not necessarily distinct) $(m_1, w_1), (m_2, w_2), \ldots, (m_k, w_k) \in M \times W$ that includes (m, w) and (m', w'),

and is such that $(m_1, w_1) \sim_{(\pi,R)} (m_2, w_2) \sim_{(\pi,R)} \ldots \sim_{(\pi,R)} (m_k, w_k) \sim_{(\pi,R)} (m_1, w_1)$. Note that $C_{\pi,R}$ is an equivalence relation on $M \times W$, i.e. it is reflexive, symmetric, and transitive. For each pair $(m, w) \in M \times W$, let $[m, w]_{C_{\pi,R}} \equiv \{(m', w') \in M \times W : (m, w)\ C_{\pi,R}\ (m', w')\}$ denote the equivalence class of (m, w) relative to $C_{\pi,R}$. Let $\gg_{(\pi,R)}$ be the relation defined on the set of all equivalence classes of $C_{(\pi,R)}$ as follows: For each pair $(m_1, w_1), (m_2, w_2) \in M \times W$, $[m_1, w_1] \gg_{(\pi,R)} [m_2, w_2]$ if and only if $[m_1, w_1] \neq [m_2, w_2]$ and there are $(m'_1, w'_1) \in [m_1, w_1]$, $(m'_2, w'_2) \in [m_2, w_2]$ such that $(m'_1, w'_1) \succeq_{(\pi,R)} (m'_2, w'_2)$. Note that, if $\pi \in P^{sd}(R)$, then $\gg_{(\pi,R)}$ is acyclic.

3.2 An efficiency theorem

In this section we show that for each probabilistic assignment that is sd-efficient at a given preference profile, there is a utility profile consistent with the ordinal preference profile such that the probabilistic assignment maximizes the sum of the expected utilities. First, let us consider the simple case where the sd-efficient assignment for which a utility function is to be constructed is an efficient deterministic assignment, μ. Let u be a utility profile consistent with R such that for each $(m, w) \in M \times W$, if m and w are matched at μ, then $u_m(w) = u_w(m) = 1$. Further, for each $w' \in W$ such that $w \succ_m w'$, let $0 < u_m(w') < \delta$ for some $\delta > 0$ and for each $w'' \in W$ such that $w'' \succ_m w$, let $1 < u_m(w'') < 1 + \epsilon$ for some $\epsilon > 0$. Let $u_w(.)$ be similarly defined. Note that for some small enough selection of δ and ϵ, the efficient assignment μ is a welfare maximizing assignment at utility profile u.

This construction would fail even for the simplest probabilistic assignment, which is obtained as a mixture of two efficient deterministic assignments. However, we show that the same conclusion holds for each sd-efficient probabilistic assignment π by providing an explicit construction for the utility profile at which π is welfare maximizing. The construction in the next example is instructive to understand the general construction to follow in Theorem 1.

Example 1. *Let $M = \{1, 2, 3\}$ and $W = \{a, b, c\}$. Let the preference profile R be as follows:*

R_1	R_2	R_3	R_a	R_b	R_c
a	b	c	3	1	2
c	a	b	2	3	1
b	c	a	1	2	3

Let μ be the assignment where 1 is matched with a, 2 with b, and 3 with c. Let μ' be the assignment where 1 is matched with b, 2 with c, and 3 with a. Let π assign 0.5 probability to each of μ and μ'.

Now, let u be such that each agent gets utility 1 from his/her top-ranked agent and utility 0 from his/her third-ranked agent. Further, each agent gets utility in the open interval $(0, 1/2)$ from his/her second-ranked agent. Clearly, for each pair (m, w), if $\pi_{mw} > 0$, then $u_m(w) + u_w(m) = 1$. Moreover, if $\pi_{mw} = 0$, then $u_m(w) + u_w(m) < 1$. Hence, the sum of expected utilities, which is 3 at π, can not exceed 3 at any probabilistic assignment.

Note that in the above construction, for each man-woman pair the sum of the utilities they get from each other is the same for each pair, and that aspect plays the critical role. For our general result a similar construction works.

Theorem 1. *For each $R \in \mathcal{R}$ and each $\pi \in P^{sd}(R)$, there is a utility profile u consistent with R such that π is ex-ante utilitarian welfare maximizing at u.*

Proof. Let $(\pi, R) \in \Pi \times \mathcal{R}$ be such that $\pi \in P^{sd}(R)$. Since we fix (π, R) throughout the proof, we remove the reference to (π, R) in denoting the binary relations we have defined, and simply write $\sim$, $\succ$, $\succeq$, and $\gg$.

Step 1: **Constructing an auxiliary utility profile.** For each $m \in M$, let $v_m : W \to \mathbb{R}$ be defined as follows. For each $w \in W$, let s_{mw} denote the length of the longest path of $\gg$ starting at $[m, w]$, and let e_{mw} denote the length of the longest path of $\gg$ ending at $[m, w]$.[13] For each $w \in W$, let

$$v_m(w) = \frac{e_{mw}}{e_{mw} + s_{mw}}$$

For each $w \in W$, we define a utility function $v_w : M \to \mathbb{R}$ in a symmetric way. For each $m \in M$, let

[13] A path of length k of $\gg$ consists of k pairs $(m_1, w_1), (m_2, w_2), \ldots, (m_k, w_k) \in M \times W$ such that $(m_1, w_1) \gg (m_2, w_2) \gg \ldots \gg (m_k, w_k)$.

$$v_w(m) = \frac{s_{mw}}{e_{mw} + s_{mw}}$$

Note that for each pair $(m, w) \in M \times W$, $v_m(w) + v_w(m) = 1$.

Lemma 1. *Let $(m, w), (m', w') \in M \times W$. If $[m, w] \gg [m', w']$, then $v_w(m) > v_{w'}(m')$.*

Proof. Suppose that $[m, w] \gg [m', w']$. Since $e_{mw} < e_{m'w'}$ and $s_{mw} > s_{m'w'}$, then

$$\frac{s_{mw}}{s_{mw} + e_{mw}} > \frac{s_{m'w'}}{s_{m'w'} + e_{m'w'}}, \text{ i.e. } \frac{1}{1 + \frac{e_{mw}}{s_{mw}}} > \frac{1}{1 + \frac{e_{m'w'}}{s_{m'w'}}}.$$

□

One consequence of the Lemma is that, for each pair $(m, w), (m', w') \in M \times W$, if $[m, w] \gg [m', w']$, then $v_m(w) + v_{w'}(m') < 1$. Let $z((m, w), (m', w')) = 1 - v_m(w) - v_{w'}(m')$ if $[m, w] \gg [m', w']$, and 1 otherwise. Let

$$\min_{(m,w),(m',w') \in M \times W} z((m, w), (m', w')) \equiv 2\epsilon.$$

Step 2: **Defining the utility profile $\boldsymbol{u}$**. Let $m \in M$ and $w \in W$. Let $u_m(w)$ be defined as follows:

i. If $\pi_{mw} > 0$, $u_m(w) \equiv v_m(w)$.

ii. If $\pi_{mw} = 0$ and there is no $w' \in W$ such that $\pi_{mw'} > 0$, $w \; R_m \; w'$, then let $u_m(w)$ be such that $u_m(w) \leq -1$.

iii. If $\pi_{mw} = 0$ and there is $w' \in W$ such that $\pi_{mw'} > 0$, $w \; R_m \; w'$, then consider a best such w', i.e. $\pi_{mw'} > 0$, $w \; R_m \; w'$, and there is no such $w'' \in W$ with $w'' \; P_m \; w'$. Let $u_m(w) \in [v_m(w'), v_m(w') + \epsilon]$.

For each $w \in W$, we define $u_w(.)$ in a symmetric way.

Step 3: **The utility profile $\boldsymbol{u}$ is consistent with $\boldsymbol{R}$**. Obviously, for each $m \in M$ (similarly for each $w \in W$), in the above construction of $u_m(.)$ the utilities in parts *ii* and *iii* can be chosen such that for each $w, w' \in W$ with $\pi_{mw} = 0$ or $\pi_{mw'} = 0$, $u_m(w) \geq u_m(w')$ if and only if $w \; R_m \; w'$. Next, we will show that for each $w, w' \in W$ with $\pi_{mw} > 0$ and $\pi_{mw'} > 0$, we have $u_m(w) \geq u_m(w')$ if and only if $w \; R_m \; w'$. Let $m \in M$

and $w, w' \in W$. Suppose that $\pi_{mw}, \pi_{mw'} > 0$. Without loss of generality, suppose that $w \; R_m \; w'$. If $w \; I_m \; w'$, then note that $(m, w') \sim (m, w)$, and $[m, w'] \sim [m, w]$. Thus, $u_m(w) = u_m(w')$ as desired. If $w \; P_m \; w'$, then note that $(m, w') \succ (m, w)$, and $[m, w'] \gg [m, w]$. Thus, $u_m(w) > u_m(w')$, as desired. For each $w \in W$, the consistency of u_w with R_w follows by symmetric arguments.

Let the function $SW(u, .) : \Pi \to \mathbb{R}$ be defined as, for each $\pi' \in \Pi$,

$$SW(u, \pi') = \sum_{(m,w) \in M \times W} [\pi'_{mw}(u_m(w) + u_w(m))].$$

Step 4: ***SW* attains its maximum at $\boldsymbol{\pi}$**. Note that for each pair $(m, w) \in M \times W$, if $\pi_{mw} > 0$, then $u_m(w) + u_w(m) = 1$. Thus, $SW(u, \pi) = n$. We will show that for each pair $(m, w) \in M \times W$, if $\pi_{mw} = 0$, then $u_m(w) + u_w(m) \leq 1$, which will imply that the maximal possible ex-ante utilitarian social welfare is n, and it is reached at π.

Let $(m, w) \in M \times W$ be such that $\pi_{mw} = 0$. Suppose that there is no $w' \in W$ such that $\pi_{mw'} > 0$, $w \; R_m \; w'$. Then, $u_m(w) \leq -1$. If there is no $m' \in M$ such that $\pi_{m'w} > 0$ and $m \; R_w \; m'$, then $u_w(m) \leq -1$ and $u_m(w) + u_w(m) < 1$. If there is $m' \in M$ such that $\pi_{m'w} > 0$ and $m \; R_w \; m'$, then $u_w(m) \leq u_w(m') + \epsilon < 2$. Thus, $u_m(w) + u_w(m) < 1$. The case when there is no $m' \in M$ such that $\pi_{m'w} > 0$, $m \; R_w \; m'$ is symmetric.

So, there is only one case left to consider. Suppose that there is $w' \in W$ such that $\pi_{mw'} > 0$ and $w \; R_m \; w'$, and there is $m' \in M$ such that $\pi_{m'w} > 0$ and $m \; R_w \; m'$. Let w' and m' be the best such agents. Note that $(m, w') \succeq (m', w)$. If $w \; I_m \; w'$ and $m \; I_w \; m'$, then by Lemma 1, $u_m(w) + u_w(m) < 1$. So, suppose that for at least one agent, the preference is strict. Then, $(m, w') \succ (m', w)$ and $[m, w'] \gg [m', w]$. Recall that $z((m, w'), (m', w)) = 1 - v_m(w') - v_w(m') \geq 2\epsilon$. Now,

$$u_m(w) + u_w(m) \leq [u_m(w') + \epsilon] + [u_w(m') + \epsilon] = v_m(w') + v_w(m') + 2\epsilon$$

which implies $u_m(w) + u_w(m) \leq 1$, as desired. □

The welfare theorem by McLennan [18] for the problem of allocating objects is a corollary of Theorem 1. To see this, first consider the following counterparts of sd-efficiency and utilitarian social welfare for that model. Let us keep the men-women notation. An assignment $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ **men-side sd-dominates** $\boldsymbol{\pi'} \in \boldsymbol{\Pi}$ **at** $\boldsymbol{R} \in \boldsymbol{\mathcal{R}}$ if for each agent $i \in M$, the lottery assigned to i at π sd-dominates the one assigned at π'.

That is, for each pair $(m, w) \in M \times W$,

$$\sum_{w':w' R_m w} \pi_{mw'} \geq \sum_{w':w' R_m w} \pi'_{mw'}$$

such that for at least one pair the inequality is strict. An assignment $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ **is men-side sd-efficient at** $\boldsymbol{R} \in \boldsymbol{\mathcal{R}}$ if no probabilistic assignment men-side sd-dominates it at R. For each utility profile $u = (u_i(.))_{i \in N}$ and probabilistic assignment π, the **men-side utilitarian social welfare at** $(\boldsymbol{u}, \boldsymbol{\pi})$ is the sum of the utilities of the men, that is:

$$MSW(u, \pi) = \sum_{m \in M} \sum_{w \in W} \pi_{mw} u_m(w)$$

A probabilistic assignment $\boldsymbol{\pi}$ **is ex-ante men-side utilitarian welfare maximizing at a utility profile** $\boldsymbol{u}$ if it maximizes the men-side social welfare at u, i.e. $\pi \in \text{argmax}_{\pi \in \Pi} MSW(u, \pi)$.

Corollary 1 (McLennan, 2002)**.** *For each $R \in \mathcal{R}$ and each $\pi \in \Pi$ that is men-side sd-efficient at R, there is a utility profile u consistent with R such that π is ex-ante men-side utilitarian welfare maximizing at u.*

Proof. Let $\pi \in \Pi$ be men-side sd-efficient at $R \in \mathcal{R}$. Let $R' \in \mathcal{R}$ be such that for each $m \in M$, $R'_m = R_m$, and each woman is indifferent between any two different men at R'. Note that π is men-side sd-efficient also at R'. Moreover, $\pi \in P^{sd}(R')$. By Theorem 1, there is a utility profile u consistent with R' such that π is ex-ante utilitarian welfare maximizing at u. Since each woman gets the same utility from any two different men at u, π is ex-ante men-side utilitarian welfare maximizing at u. Now, let u' be a utility profile such that for each $m \in M$, $u_m(.) = u'_m(.)$, and for each $w \in W$, $u'_w(.)$ is consistent with R_w. Note that π is ex-ante men-side utilitarian welfare maximizing at u'. □

3.3 A stability theorem

A central robustness criterion for deterministic assignments is "stability", which requires that there is no unmatched man-woman pair who prefer each other to their assigned mates. A counterpart of stability for probabilistic assignments is "ex-post" stability,

which requires that there is at least one decomposition of the probabilistic assignment into stable deterministic assignments.

Let $\mathcal{D}$ denote the set of deterministic assignments. An assignment $\boldsymbol{\mu} \in \boldsymbol{\mathcal{D}}$ **is stable at** $\boldsymbol{R} \in \boldsymbol{\mathcal{R}}$ if there is no $(m, w) \in M \times W$ such that $m \; P_w \; \mu(w)$, $w \; P_m \; \mu(m)$. An assignment $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ **is ex-post stable** if it can be expressed as a convex combination of stable deterministic assignments.

From this point on, we restrict ourselves to strict preferences. For each $i \in N$, let $\mathcal{P}_i \subset \mathcal{R}_i$ be the set of all transitive, anti-symmetric, and complete preference relations for i. Let $\mathcal{P} = \times_{i \in N} \mathcal{P}_i$ be the set of all strict preference profiles.

We show that ex-post stability imply welfare properties, beyond sd-efficiency, which are also deducible from the ordinal preferences and can avoid ex-ante break aways of men-women coalitions from the society.

A coalition $S = M' \cup W' \subseteq M \times W$ is **admissible** if $|M'| = |W'|$. Let $\mathcal{A}$ be the **set of all admissible coalitions**. For each $S \in \mathcal{A}$, let Π^S denote the set of probabilistic assignments defined over S. For each utility profile $u = (u_i)_{i \in N}$, let V^u be the **transferable utility game** such that for each $S = M' \cup W' \in \mathcal{A}$, $V^u(S)$ is the maximum total expected utility coalition S can achieve among its members. That is, for each $S = M' \cup W' \in \mathcal{A}$,

$$V^u(S) = \max_{\pi^S \in \Pi^S} \sum_{(m,w) \in M' \times W'} \pi^S_{mw}(u_m(w) + u_w(m)).$$

Let $E(u_m|\pi) = \sum_{w \in W} \pi_{mw} u_m(w)$ and $E(u_w|\pi) = \sum_{m \in M} \pi_{mw} u_w(m)$ be the expected value of u_m at π and the expected value of u_w at π, respectively. Given a utility profile u, an assignment $\pi \in \Pi$ is in the **core of** $\boldsymbol{V^u}$ if no coalition can increase its total expected utility by deviating to another probabilistic assignment where they are matched among themselves. That is, for each $S \in \mathcal{A}$,

$$V^u(S) \leq \sum_{m \in M'} E(u_m|\pi) + \sum_{w \in W} E(u_w|\pi).$$

Let $C(V^u)$ be the **set of all assignments that are in the core of** $\boldsymbol{V^u}$.

Let $P \in \mathcal{P}$. Let P_M and P_W denote the common preferences of men and women over deterministic assignments induced by P, defined as follows: For each $\mu, \mu' \in M$, $\mu \; P_M \; \mu'$ if and only if for each $m \in M$, $\mu \; P_m \; \mu'$. The relation P_W is defined similarly.

An assignment $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ **is well-ordered ex-post stable at** $\boldsymbol{P} \in \boldsymbol{\mathcal{P}}$ if it has a decomposition into stable assignments $\mu_1, \ldots, \mu_T$ such that for each $t, t' \in \{1, \ldots, T\}$ with $t < t'$, we have $\mu_t \; P_M \; \mu_{t'}$ and $\mu_{t'} \; P_W \; \mu_t$.

Proposition 2. *Let $P \in \mathcal{P}$, $\pi \in \Pi$. If π is a well-ordered ex-post stable assignment, then there is a utility profile u consistent with P such that $\pi \in C(V^u)$.*

Proof. Suppose that π has the following decomposition into stable assignments: $\pi = \lambda_1\mu_1 + \lambda_2\mu_2 + \cdots + \lambda_T\mu_T$. Suppose that for each $t, t' \in \{1, 2, \ldots, T\}$ such that $t < t'$, $\mu_t \; P_M \; \mu_{t'}$ and $\mu_{t'} \; P_W \; \mu_t$.

We first define the utilities each agent gets from the agents that he/she is matched with with positive probability. Let $(m, w) \in M \times W$ such that $\pi_{mw} > 0$. First note that if m is matched to w in two different assignments in the decomposition, say $\mu_t, \mu_{t'}$, $t < t'$, then they should be matched in all the assignments between μ_t and $\mu_{t'}$, i.e. for each $t'' \in \{t, t+1, \ldots, t'\}$, $\mu_{t''}(m) = w$. So, let $\mu_p, \mu_{p+1}, \ldots \mu_q$ be all the assignments at which m is matched to w. Let $\lambda_{mw} \equiv \lambda_p + \lambda_{p+1} + \cdots + \lambda_q$. Let

$$u_w(m) = \frac{\lambda_p}{\lambda_{mw}} \cdot \frac{p}{T+1} + \frac{\lambda_{p+1}}{\lambda_{mw}} \cdot \frac{p+1}{T+1} + \cdots + \frac{\lambda_q}{\lambda_{mw}} \cdot \frac{q}{T+1}.$$

Let

$$u_m(w) = \frac{\lambda_p}{\lambda_{mw}} \cdot \frac{T-p+1}{T+1} + \frac{\lambda_{p+1}}{\lambda_{mw}} \cdot \frac{T-p}{T+1} + \cdots + \frac{\lambda_q}{\lambda_{mw}} \cdot \frac{T-q+1}{T+1}.$$

Note that if there is a unique assignment μ_t in the decomposition such that $\mu_t(m) = w$, we simply have $u_w(m) = \frac{t}{T+1}$ and $u_m(w) = \frac{T-t+1}{T+1}$.[14]

Next, we argue that the utilities each agent gets from the agents that he/she is matched with with zero probability can be defined in such a way that for each such pair $(m, w) \in M \times W$, $u_m(w) + u_w(m) < 1$. Let $(m, w) \in M \times W$ be such that $\pi_{mw} = 0$. Note that if there is no $w' \in W$ such that $\pi_{mw'} > 0$ and $w \; P_m \; w'$, then let $u_m(w) \leq -1$. The case when there is no $m' \in W$ such that $\pi_{m'w} > 0$, $m \; P_w \; m'$ is the same. So, suppose that there are $m' \in M, w' \in W$ such that $\pi_{mw'} > 0$, $w \; P_m \; w'$, and $\pi_{m'w} > 0$, $m \; P_w \; m'$. Suppose w.l.o.g. that w' and m' are best such agents at P_w and P_m. We will show that $u_m(w') + u_w(m') < 1$. First, note that the pairs (m, w') and (m', w) can not be matched pairs in the same assignment of the decomposition, since otherwise that assignment would not be stable. Suppose that m and w' are

[14] In this case, the utility function is exactly the one constructed in Theorem 1.

matched in assignments $\mu_p, \mu_{p+1}, \ldots, \mu_q$, and m' and w are matched in assignments $\mu_{p'}, \mu_{p'+1}, \ldots, \mu_{q'}$. Note that either $q < p'$ or $q' < p$. Observe that we can not have $q < p'$. Otherwise, m would prefer his mate in μ_q, namely w', to his mate in $\mu_{p'}$. But then, m would prefer w to his mate in $\mu_{p'}$, contradicting the assumption that $\mu_{p'}$ is stable. Thus $q' < p$. Note that $u_m(w') \leq \frac{T-p+1}{T+1}$ and $u_w(m') \leq \frac{q'}{T+1}$. Thus, $u_m(w') + u_w(m') < 1$. Then, by arguments similar to the proof of Theorem 1, for each such m, w, m, w', let $z((m,w),(m',w')) = 1 - u_m(w) - u_{w'}(m')$ and let 2ϵ be the minimum of $z((m,w),(m',w'))$. Now, for the pair (m,w), let $u_m(w) \in [u_m(w'), u_m(w') + \epsilon]$ and let $u_w(m) \in [u_w(m'), u_w(m') + \epsilon]$. Thus, utility profile u is constructed such that it is consistent with P and for each pair (m,w) that is matched with zero probability, we have $u_m(w) + u_w(m) < 1$.

Now, we will show that indeed $\pi \in C(V^u)$. First, note that for each admissible coalition $S = M' \cup W'$, $V^u(S) \leq |M'|$. Now, let $(m,w) \in M \times W$. We will show that $E(u_m|\pi) + E(u_w|\pi) = 1$. Observe that;

$$E(u_m|\pi) = \lambda_1 \frac{T}{T+1} + \lambda_2 \frac{T-1}{T+1} + \cdots + \lambda_T \frac{1}{T+1}.$$

Also observe that

$$E(u_w|\pi) = \lambda_1 \frac{1}{T+1} + \lambda_2 \frac{2}{T+1} + \cdots + \lambda_T \frac{T}{T+1}.$$

Then, $E(u_m|\pi) + E(u_w|\pi) = \lambda_1 + \lambda_2 + \cdots + \lambda_T = 1$. Thus, for each admissible coalition $S = M' \cup W'$,

$$\sum_{i \in S} E(u_i|\pi) = |M'| \geq V^u(S)$$

Hence, $\pi \in C(V^u)$.

□

Next, we show that well-ordered ex-post stability is equivalent to ex-post stability. The proof relies on the following lemma. The first part of the lemma is due to Rothblum [22] and the second part is due to Roth et al. [20].

Lemma 2. *Let $P \in \mathcal{P}$ and $\pi \in \Pi$. Let π be ex-post stable at P.*

1. *For each pair* $(m, w) \in M \times W$,

$$\pi_{mw} + \sum_{w' : w P_m w'} \pi_{mw'} + \sum_{m' : m P_w m'} \pi_{m'w} \leq 1.$$

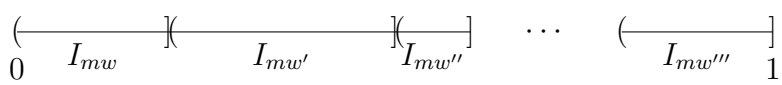
2. *For each pair* $(m, w) \in M \times W$ *such that* $\pi_{mw} > 0$,

$$\pi_{mw} + \sum_{w' : w P_m w'} \pi_{mw'} + \sum_{m' : m P_w m'} \pi_{m'w} = 1.$$

Proposition 3. *If* $\pi \in \Pi$ *is ex-post stable at* $P \in \mathcal{P}$*, then* π *has a well-ordered stable decomposition.*

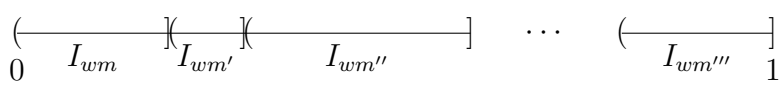
Proof. Let π be ex-post stable at P. We will construct a well-ordered stable decomposition of π using the rounding approach due to Sethuraman and Teo [24].

For each $m \in M$, let $\pi_m^+ \equiv \{w \in W : \pi_{mw} > 0\}$. We partition the $(0, 1]$ interval into $|\pi_m^+|$ intervals $(I_{mw})_{w \in \pi_m^+}$ such that each interval I_{mw} is of the type $(a_{mw}, b_{mw}]$ (left-open, right-closed) with length π_{mw}, and if $w \; P_m \; w'$, then $a_{mw} < a_{mw'}$.



$$w \; P_m \; w' \; P_m \; w'' \; P_m \; w'''$$

For each $w \in W$, let $\pi_w^+ \equiv \{m \in M : \pi_{mw} > 0\}$. We partition the $(0, 1]$ interval into $|\pi_w^+|$ intervals $(I_{wm})_{m \in \pi_w^+}$ such that each interval I_{wm} is of the type $(a_{wm}, b_{wm}]$ (left-open, right-closed) with length π_{mw}, and if $m \; P_w \; m'$, then $a_{wm} > a_{wm'}$.



$$m''' \; P_w \; m'' \; P_w \; m' \; P_w \; m$$

Due to Lemma 2 Part 2, for each pair (m, w), the intervals I_{mw} and I_{wm} coincide. Thus, the following procedure gives a well-defined matching: Pick a real number $x \in (0, 1]$. For each pair (m, w), match m with w if and only if $\pi_{mw} > 0$ and $x \in I_{mw}$ [or equivalently $x \in I_{wm}$]. Also, this matching is stable: for each (m, w) such that m prefers w to his match, either $\pi_{mw} > 0$ and I_{mw} is on the left of x and therefore I_{wm} is on the left of x and w does not prefer m to her match, or $\pi_{mw} = 0$ and due

to Lemma 2 Part 1, w does not prefer m to her match. Now, if the real number x is picked according to the Uniform distribution on $(0, 1]$, it is easy to see that we have a well-ordered decomposition of π. □

Theorem 2. *Let $P \in \mathcal{P}$, $\pi \in \Pi$. If π is ex-post stable, then there is a utility profile u consistent with P such that $\pi \in C(V^u)$.*

Proof. Follows from Propositions 2 and 3. □

The following example shows that the converse of Theorem 2 is not true.

Example 2. *Let $M = \{1, 2, 3\}$ and $W = \{a, b, c\}$. Let the preference profile R and the utility profile u that is consistent with R be as follows.*

R_1	u_1	R_2	u_2	R_3	u_3
a	$\frac{2}{3}$	a	$\frac{2}{3}$	a	$\frac{1}{2}+\epsilon$
c	$\frac{1}{3}$	c	$\frac{1}{3}$	b	$\frac{1}{2}$
b	0	b	0	c	0

R_a	u_a	R_b	u_b	R_c	u_c
1	$\frac{2}{3}$	3	$\frac{1}{2}$	2	$\frac{2}{3}$
3	$\frac{1}{3}+\epsilon$	1	$\frac{1}{3}+\epsilon$	3	$\frac{1}{3}+\epsilon$
2	$\frac{1}{3}$	2	$\frac{1}{3}$	1	$\frac{1}{3}$

Let μ be the assignment where 1 is matched with a, 2 with c, and 3 with b. Let μ' be the assignment where 1 is matched with c, 2 with a, and 3 with b. Note that $(3, a)$ forms a blocking pair in μ'. Let π be the assignment that assigns 0.5 probability to each of μ and μ'. Since this is the only possible decomposition of π, it is not ex-post stable.

Next, let u be as described with the numeric values in the profile. Note that each man or woman receives the same expected utility $1/2$ from π at u. Moreover, for each pair $(m, w) \in M \times W$, if $\pi_{mw} > 0$ then $u_m(w) + u_w(m) = 1$ and for a small enough choice of ϵ, if $\pi_{mw} = 0$ then $u_m(w) + u_w(m) < 1$. It follows that $\pi \in C(V^u)$.

For deterministic assignments, if we restrict ourselves to strict preferences (at which no agent is indifferent between any two different agents), stability implies efficiency. To see this, suppose that a deterministic assignment is not efficient. Then there is another assignment at which an agent, say agent i, is better off, which means that i is matched to agent j whom he prefers to his/her current mate. Since j can not be worse off in the new assignment, j prefers i to his/her current mate. Thus, the initial assignment can

not be stable. However, once indifferences are allowed, there is no implication relation between efficiency and stability.[15]

As opposed to the deterministic case, in the probabilistic case, the relation between sd-efficiency and ex-post stability is not evident. If we allow for indifferences, since efficiency is not related to stability in the deterministic case, ex-post stability and sd-efficiency are not related either. However, if we restrict ourselves to strict preferences, the relation between ex-post stability and sd-efficiency is not so clear. Yet, it follows from Theorem 2 that ex-post stability implies sd-efficiency when preferences are strict.[16]

Corollary 2. *If $\pi \in \Pi$ is ex-post stable at $P \in \mathcal{P}$, then π is sd-efficient at P.*

In a recent study, Echenique and Galichon [10] question if there is a utility profile consistent with the ordinal preferences such that each stable deterministic assignment is in the core of the associated TU-game. They provide the following partial answer to this question: Define an *isolated stable assignment* as a stable deterministic assignment such that for each pair of matched agents, there is no other stable assignment at which they are matched to each other; then, there is a utility profile consistent with the ordinal preferences such that each isolated stable assignment is in the core. To see that this result also follows from our results, consider the ex-post stable assignment that is a uniform lottery over these isolated stable deterministic assignments. It is easy to see that this lottery is the unique decomposition of the ex-post stable assignment into stable deterministic assignments, and therefore the isolated stable deterministic assignments must be well-ordered. Now, one can show that the utility profile that follows from our construction is such that not only the ex-post stable probabilistic assignment, but also each of these deterministic isolated stable assignments belongs to the core. It is simply due to the fact that for each isolated assignment, for each unmatched man-woman pair, the sum of the utilities the man gets from his mate and the utility the woman gets

[15]Consider a serial dictatorship, which, according to a predetermined order, assigns each man to one of his most preferred women from among the remaining ones. One can easily specify preferences so that this produces an efficient but unstable assignment. To see that there is a stable but inefficient assignment, consider two men $\{m_1, m_2\}$ and two women $\{w_1, w_2\}$. Let the preferences be such that each woman is indifferent between the two men, m_1 prefers w_1 to w_2, and m_2 prefers w_2 to w_1. Note that assigning m_1 to w_2, and m_2 to w_1 is stable but inefficient.

[16]To see this note that for a probabilistic assignment π, if there is a utility profile u such that $\pi \in C(V^u)$, then π is utilitarian efficient at u. This result is independently reported also by Manjunath [16] (Proposition 2), the proof of which relies on Lemma 2 (Roth et al. [20]) that we report in section 3.4.

from her mate is equal to 1. As Echenique and Galichon [10] point out, it is not clear if this result can be generalized to the entire set of stable assignments. However, it is rather clear that one can construct a utility profile such that each stable assignment is utilitarian welfare-maximizing.

4 Extensions of the Model

We present two extensions of our model and discuss whether our results still follow.

One-to-many setting: Suppose that there is a set of workers, denoted by W, and a set of firms, denoted by F. Each firm $f \in F$ has a capacity c_f such that $\sum_{f \in F} c_f = |W|$. Each worker has a strict preference relation over firms and vice versa. In a deterministic assignment, each worker is assigned to a firm. A probabilistic assignment is an $|W| \times |F|$ stochastic matrix where each row corresponds to a worker, each column corresponds to a firm, each row sum is equal to 1, and each column sum is equal to the capacity of the corresponding firm (Birkhoff [4] and von Neumann [27] extends to this setting). A deterministic assignment is ex-post stable if there is no worker-firm pair such that the worker prefers the firm to his assigned firm, and the firm prefers the worker to one of his assigned workers. A probabilistic assignment is ex-post stable, if it can be decomposed into stable deterministic assignments. We will argue that our core result still follows: for each ex-post stable assignment, there is a utility profile $u = (u_i)_{i \in W \cup F}$ such that no group of workers and firms (such that the total capacity is equal to the total number of workers) can match among themselves and achieve a higher total expected utility (the utility of a firm is the sum of the utilities it gets from different workers).

Consider an ex-post stable assignment π with a stable decomposition $\pi = \lambda_1 \mu_1 + \lambda_2 \mu_2 + \cdots + \lambda_T \mu_T$. Consider the following auxiliary one-to-one problem and the auxiliary assignment π': for each firm f, there are $|c_f|$ copies, indexed from 1 to $|c_f|$, such that each copy has the same preference relation as the firm's original preference relation; each worker's preference relation is such that given two copies of firms, if the copies belong to different firms, then the worker's preference relation agrees with his original preference relation, and if the two copies belong to the same firm, then the worker prefers the copy with the lower index to the other copy. Note that the auxiliary problem belongs to our one-to-one setting. For each μ_t where $t \in \{1, \ldots, T\}$, let μ'_t be the deterministic assignment in the one-to-one setting with copies such that each worker is assigned to a

copy of the firm that he is assigned to at μ_t, and for each pair of copies of the same firm, if the copy with a higher index is assigned a worker, then the copy with a lower index is assigned a more preferable worker. Let $\pi' = \lambda_1\mu'_1 + \lambda_2\mu'_2 + \cdots + \lambda_T\mu'_T$. It is easy to see that π' is ex-post stable at the auxiliary one-to-one problem. Then, by Theorem 2, there is a utility profile u' such that π' is in the core. Note that two copies of the same firm may have different utility functions; similarly, workers may have different utilities over the copies of the same firm.

Next, we argue that we can aggregate the utilities to a single utility both for the firms and for the workers such that the core result follows for the one-to-many setting. Suppose that π' has a following well-ordered decomposition into the stable assignments $(\mu''_1, \mu''_2, \ldots, \mu''_K)$. Let w be a worker and f_i be the ith copy of firm f. Note that, whether they are matched or unmatched, we have, $u'_w(f_i) + u'_{f_i}(w) \leq 1$, and $E_w(u'|\pi') + E_{f_i}(u'|\pi') = 1$. Now, let I_{fw} be the set of the copies of firm f that are matched to w with positive probability at π'. If $I_{fw} = \emptyset$, then take any copy f_i and define $u_f(w) = u'_{f_i}(w)$ and $u_w(f) = u'_w(f_i)$. If $I_{fw} \neq \emptyset$, then let $\lambda_{fw} = \sum_{i \in I_{fw}} \pi'_{f_i w}$, and define

$$u_f(w) = \sum_{i \in I_{fw}} \frac{\pi'_{f_i w}}{\lambda_{fw}} \cdot u'_{f_i}(w),$$

$$u_w(f) = \sum_{i \in I_{fw}} \frac{\pi'_{f_i w}}{\lambda_{fw}} \cdot u'_w(f_i).$$

The utility profile u is consistent with the ordinal preference profile at the one-to-many problem. To see this, note that if w is assigned to two different copies of the firm f in two different members of the decomposition, say $\mu''_t, \mu''_{t'}$ such that $t < t'$, then in all the assignments between μ''_t and $\mu''_{t'}$, w is matched with a copy of firm f.

Next, we argue that the initial one-to-many assignment π is in the core of the TU-game associated with utility profile u. The proof of this argument is vis-à-vis the proof of the same argument for Theorem 2. By our construction of u, the value of a coalition can not exceed the number of workers (equivalently the total capacity of the firms) in the coalition, which is exactly the sum of the expected utilities of the coalition members.

Allowing for remaining single: Consider the extension of our model where each man or woman is allowed to remain single, and the preference relation of each $m \in M$ (similarly for each $w \in W$) is a strict preference relation over $W \cup \{m\}$, where m

represents remaining single.

For our results related to sd-efficiency, for each $m \in M$ and $w \in W$ add the pairs (m, m) and (w, w) into the binary relation that we defined to characterize sd-efficiency. One can easily see that in this extended setting sd-efficiency of a probabilistic assignment is equivalent to the acyclicity of this extended relation. Since for Theorem 1 our construction of the utility profile is based on this binary relation, the construction of the utility profile for this extended case follows similarly in which for each agent we also obtain a utility index for remaining single.

As for the extension of Theorem 2, the key is to note that: for a given preference profile, by the Rural Hospital Theorem (Roth [19]), at each stable matching, the same men and women are single. It follows that each ex-post stable probabilistic assignment induces a sub-matrix which is doubly-stochastic, and some men and women that always remain single. Now, for the matched men and women, consider the utility profile obtained by our Theorem 2. The only question is how to incorporate the utility of remaining single to this profile. Since the given assignment is ex-post stable, each agent prefers each other agent that he or she is matched with positive probability to remain single. So, for each agent, let the utility he/she gets from remaining single be -1. One can construct the rest of the utility profile such that for each unmatched man-woman pair, the utilities they get from each other are less than 0. Hence, remaining single or the possibility of getting matched with agents who are single is not a threat for the given ex-post assignment to be in the core.

5 Conclusion

Marriage problems constitute a basis for many real-life economic problems. Here, we considered the efficiency and stability of probabilistic assignments in marriage problems when only ordinal preference information is available. When we only have ordinal preference information, two common ways to evaluate probabilistic assignments in terms of efficiency and stability are sd-efficiency and ex-post stability, respectively. We asked whether probabilistic assignments that are sd-efficient or ex-post stable for the ordinal preferences are possibly efficient or stable for the cardinal preferences. Our answer was positive. Of course, this positive answer does not provide a strong justification for the efficiency and stability notions mentioned above, as there may be several utility

profiles consistent with the ordinal preferences for which the sd-efficient or ex-post stable assignments are not desirable. However, our results show that there is at least one utility profile for which these assignments continue to be desirable, which provides a further understanding of the structure of such assignments and the connection between efficiency and stability notions in the ordinal and cardinal environments.

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