Two-stage Rights Structures and Nash Bargaining

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ABSTRACT: Koray & Yildiz (2018) introduces a new framework for implementation in which the main tool to design is the *rights structure* introduced by Sertel (2001). It was assumed that there is only one stage to obtain the equilibrium outcome of a rights structure. We formulate implementation via two-stage rights structures and show that the *Nash bargaining solution* is implementable via two-stage rights structures.

KEYWORDS: Rights structures, Nash bargaining, multi-stage implementation.
AEA classification: D0, C0.

This paper subsumes and extends Yıldız (2013)[Section 1.7]. I am grateful to Ariel Rubinstein for long discussions on the subject, and especially for bringing Rubinstein, Safra & Thomson (1992) and Rubinstein (1995) to my awareness.

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1. Introduction

In mechanism design, one of the major concerns is whether the framework within which the agents are interacting with each other is sufficiently simple and familiar. In this vein, a persistent criticism of implementation theory has been that the game forms constructed to obtain general results have “unnatural" features that take away from the relevance of the theory.[1] This calls for a new framework that is formulated in a language closer to that of real life mechanisms. With this motivation, Koray & Yildiz (2018) proposed a new framework for implementation in which the main tool to design is the *rights structure* introduced by Sertel (2001). Although, Koray & Yildiz (2018) covers a rather extensive treatment of the subject, it is assumed that there is only one stage to obtain the equilibrium outcome of a rights structure. In this short paper, we formulate implementation via two-stage rights structures. Then, to demonstrate that this formulation can be fruitful for implemention, we show that the *Nash bargaining solution* is implementable via an intuitive two-stage rights structures that mimics Rubinstein, Safra & Thomson (1992)'s reformulation of the *Nash bargaining solution* in natural language.

Following Koray & Yildiz (2018) (also referred to as KY from now on), a *rights structure*, consists of a state space S, an outcome function h – which associates an alternative with each state – and a code of rights γ. A code of rights γ assigns to each pair of distinct states (s,t) the family $s \xrightarrow{\gamma} t$ of coalitions that are entitled to approve a change from state s to state t. Thus, an existing state s will be converted to another state t only if there is some coalition, which is entitled to change s to t and is willing to do so. If there is no state t different than s along with such a coalition, then s is an equilibrium state under γ and the preference profile $\succsim$ of the society.

A two stage rights structures, denoted by Γ^2, is a quadruple (S,h,γ^1,γ^2). At the first stage, the rights structure $\Gamma^1=(S,h,\gamma^1)$ is active, and the non-equilibrium states according to Γ^1 are eliminated, preserving the equilibrium states $E(\Gamma^1,\succsim)$ for the next stage. At the second stage, the rights structure $(E(\Gamma^1,\succsim),h,\gamma^2)$ is active, and the associated equilibrium states form the equilibria of the two-stage rights structures Γ^2. As usual, a social

[1] Some sort of an *integer game* or *modulo game* is used to eliminate strategies with unacceptable outcomes from the equlibria. As it is argued in detail by Jackson Jackson (1992) and by Abreu and Matsushima Abreu & Matsushima (1992), besides being difficult to interpret, there are several technical problems associated with these games.

choice rule, which chooses at each preference profile, the equilibrium outcomes that are induced by the equilibrium states via the outcome function h, will be said to be implemented by the the two-stage rights structure in question.

In the formulation of a rights structures, a critical component is a "state", which can represent a resource allocation, an economic, political or legal arrangement. Further, a state can itself be a constitution or legal code that determines a social situation as well as the coalitions that are entitled to move from the current state to the others (Koray 2000, Barbera & Jackson 2004, Acemoglu, Egorov & Sonin 2012). The formulation of two-stage rights structures is particularly in line with the latter interpretation. In that, a state that arises as an equilibrium out of the first stage rights structures determines the coalitions that have rights to move to the other states

2. Implementation via Two-Stage Rights Structures

We introduce the preliminaries by closely following the terminology of Koray & Yildiz (2018). We use A to denote a nonempty set of alternatives, and N a nonempty finite set of n agents. Each nonempty subset of N is called a **coalition**, and denoted generically by K. For given A and N, for each $i \in N$, $\succsim_i$ denotes the **preference relation** of agent i, which is a *complete* and *transitive* binary relation on A. For each distinct pair $a, b \in A$, $a \succ_i b$ means that "i prefers a to b" and $a \sim_i b$ means that "i is indifferent between a and b ". A **preference profile** $\succsim$ associates each agent with a preference relation $\succsim_i$, i.e. $\succsim = [\succsim_1, \ldots, \succsim_n]$. The set of all preference profiles is denoted by $\mathscr{P}$. A **social choice rule** (SCR) F maps each preference profile into a nonempty subset of A, i.e. $F : \mathscr{P} \to 2^A \setminus \{\emptyset\}$.

A **rights structure** Γ, as used by KY, is a triplet (S, h, γ), where S is the **state space**, and h is the **outcome function** that maps each state to an alternative, i.e. $h : S \to A$. Given a state space S, a **code of rights** specifies for each pair of distinct states $s, t \in S$, a family of coalitions denoted by $s \xrightarrow{\gamma} t$. The interpretation is that: each coalition in $s \xrightarrow{\gamma} t$ is entitled to approve a change from s to t by the code of rights γ. For an additionally given outcome function h and preference profile $\succsim$, for each distinct $s, t \in S$ a coalition K prefers t to s if and only if for each $i \in K$, $h(t) \succ_i h(s)$. This is denoted by $h(t) \succ_K h(s)$.

In order to define implementation via rights structures, called Γ-implementability, first we will remind the Γ-equilibrium notion, which plays the role of the solution concepts (e.g. Nash equilibrium) in the classical implementation.

Definition 1 (Koray & Yildiz (2018)) Let $\Gamma = (S, h, \gamma)$ be a rights structure and $\succsim \in \mathscr{P}$ be a preference profile. Then, a state $s \in S$ is a Γ**-equilibrium** at $\succ$ if for each $t \in S \setminus \{s\}$ there is no coalition $K \in s \xrightarrow{\gamma} t$ with $h(t) \succ_K h(s)$.

An SCR F is Γ**-implementable** if there exists a rights structure Γ such that for each $\succsim \in \mathscr{P}$, $F(\succsim) = h(E(\Gamma, \succsim))$.

A two stage rights structures, denoted by Γ^2, is a quadruple $(S, h, \gamma^1, \gamma^2)$, which is obtained by enriching a rights structures by adding a second stage code of rights γ^2, which can be different than γ^1. A two stage rights structures can be considered as a counterpart of two-stage extensive form games. Next, we define the equilibrium notion that we propose for a two stage rights structures.

Definition 2 Let $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$ be a two-stage rights structure and $\succsim$ be a preference profile. Then, a state $s \in S$ is an **equilibrium of** Γ^2 **at** $\succsim$, denoted by $s \in E(\Gamma^2, \succsim)$, if the following two conditions hold:

1. $s \in E(S, h, \gamma^1)$, and
2. for each $t \in E(S, h, \gamma^1)$, there is no coalition $K \in s \xrightarrow{\gamma^2} t$ with $h(t) \succ_K h(s)$.

An SCR F is implementable via two-stage rights structures, called Γ^2**-implementable**, if there exists a two stage rights structures Γ^2 such that for each $\succ \in \mathscr{P}$, $F(\succsim) = E(\Gamma^2, \succsim)$.

3. Bargaining Problem and Nash Solution

We first provide the classical formulation of a bargaining problem. A *bargaining problem* consists of the *feasible set* U and a *disagreement point* d. Each element of U is a **utility** pair for two agents. The utilities are *von Neumann-Morgenstern* utilities, that is values extended to lotteries (probability measures with finite support) from deterministic allocations, by satisfying the expected utility axioms, namely *independence* and *continuity* axioms. A *bargaining solution* is a function that assigns a unique pair of utility levels to each problem $\langle U, d\rangle$, where U is a compact and convex subset of a two dimensional Euclidean space containing a point that provides a higher utility level for both agents, compared to d. Rubinstein, Safra & Thomson (1992) (also referred to as RST from now on) argue that $\langle U, d\rangle$ is a condensed form of a problem $\langle X, D, \succsim_1, \succsim_2\rangle$, where X is a set of feasible alternatives, described in **physical** terms, D is the disagreement alternative, and $\succsim_1, \succsim_2$ are the preferences defined on the space of lotteries over deterministic prizes $X \cup D$. We assume that X is a compact and convex subset of a finite dimensional Euclidean space that contains an alternative that is preferable to D for both agents.

Next, in this "alternative-preference based setting", we introduce the formulation of RSD for the Nash bargaining solution. For this, we need to introduce a piece of notation: for each $x \in X$ and $p \in [0,1]$, px stands for the lottery that yields x with probability p, and yields d with probability $1-p$.

Definition 3 (Rubinstein, Safra & Thomson (1992)) *An **ordinal-Nash solution** (outcome) for the bargaining problem $\langle X, D, \succsim_1, \succsim_2\rangle$ is an alternative y^* such that for each $p \in [0,1]$ and for each $x \in X$, if $px \succ_i y^*$, then $py^* \succsim_j x$, where $i, j \in \{1,2\}$.*

As stated by Osborne & Rubinstein (1994, Chapter 15), the structure of the ordinal-Nash solution is similar to that of many solution concepts in cooperative game theory, in which an outcome is a solution if for any valid "objection", there is a valid "counter objection". Specifically, an ordinal-Nash solution is an alternative such that every valid objection of the form "I demand the alternative x rather than y^*; I back up this demand by threatening to take steps that will cause us to fail to agree with probability $1-p$, a threat that is credible since if I carry it out, then I will be better of" can be responded by the other agent with a valid counter objection of the form "If you take steps that will cause us to disagree with probability $1-p$ then it is still desirable for me to insist on y^* rather than agreeing to x."

For a given pair $\langle X, D\rangle$, let $\succsim_1, \succsim_2$ be a pair of preferences defined on the space of lotteries over deterministic prizes $X \cup D$. Then, consider following conditions:

i. $\succsim_1$ and $\succsim_2$ are expected utility preferences, i.e. both satisfy the von Neumann-Morgenstern *independence* and *continuity* axioms.

ii. For each $x \in X$, and for both $i \in \{1,2\}$, $x \succsim_i D$ and there exists at least one $x \in X$ such that for both $i \in \{1,2\}$, $x \succ_i D$.

iii. There are no two alternatives $x, x' \in X$ such that for both $i \in \{1,2\}$, $x \sim_i x'$.

Let $\mathscr{P}$ be the set of all preference pairs that satisfy (i), (ii) and (iii). Then, it follows from Proposition 1 of RST that: for each problem $\langle X, D, \succsim_1, \succsim_2\rangle$ with $[\succsim_1, \succsim_2] \in \mathscr{P}$, the ordinal Nash solution is well defined and coincides with the unique classical (utility) Nash bargaining solution (Nash 1950). Now, by using the formulation of RSD, we can describe the **Nash bargaining solution** as the SCR F that maps each preference profile $[\succsim_1, \succsim_2] \in \mathscr{P}$, to the unique ordinal-Nash solution for the problem $\langle X, D, \succsim_1, \succsim_2\rangle$.

4. Result

We construct a two-stage rights structures that implements the Nash bargaining solution. Next, we describe this two-stage rights structures $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$.

State space: $S = X \cup \{(px, y) \mid x, y \in X \text{ and } p \in (0,1)\}$.

Outcome function: For each $x, (px, y) \in S$, $h(x) = x$ and $h(px, y) = px$.

Movements allowed by first stage code of rights: For each $x, y \in X$ and $p \in (0,1)$,

$$(px, y) \xrightarrow{\gamma^1} (p^2 y, x) = \{\{1\}, \{2\}\}.$$

Movements allowed by second stage code of rights: For each $x, y \in X$ and $p \in (0,1)$,

$$y \xrightarrow{\gamma^2} (px, y) = \{\{1\}, \{2\}\} \text{ and } (px, y) \xrightarrow{\gamma^2} x = \{\{1\}, \{2\}\}.$$

Before proving the result, we give the intuition for our choice of this specific two-stage rights structures. In line with the interpretation of the ordinal-Nash solution, a state of the form (px, y) can be interpreted as an objection to materializing the alternative y. Then, if an agent moves from (px, y) into $(p^2 y, x)$, this shows that he has a valid counter objection to materializing the alternative x. Thus, in the first stage, valid objections that have valid counter objections are eliminated. In the second stage, by using the non-eliminated valid objections, the undesired alternatives are eliminated. Finally, all the remaining valid objections are eliminated in favor of alternatives that both agents prefer, and thus the desired alternative is singled out as the unique equilibrium state.

There are two more notable aspects of the two-stage rights structures Γ^2. First,it is *individual based*, as defined by Koray & Yildiz (2018): only singletons have the right to move from a state into another. Second, it is *symmetric*, in the sense that whenever an agent has a right to move from one state to another, the other agent has the same right as well.

Proposition 1 *The Nash bargaining solution is implementable via a two stage rights structure.*

Proof. Let $\succsim = [\succsim_1, \succsim_2] \in \mathscr{P}$ and let $\{y^*\}$ be the unique ordinal-Nash solution for the problem $\langle X, D, \succsim_1, \succsim_2 \rangle$. First, we show that $y^* \in E(\Gamma^2, \succsim)$. To see this, note that in the first stage, for each $x \in X$, no one has the right to move from x. Hence, y^* can not be eliminated at the first stage. For the second stage, suppose that there exists $x \in X$, $p \in (0,1)$, and $i \in \{1,2\}$, such that $i \in y^* \xrightarrow{\gamma^2} (px, y^*)$ and $px \succ_i y^*$. Assume without loss of generality that $i = 1$. Now, by the definition of ordinal Nash solution, we must have $py^* \succsim_2 x$. Then, since $\succsim_2$ satisfies *independence*, it follows that $p^2y^* \succsim_2 px$. Since $\{2\} \in (px, y^*) \xrightarrow{\gamma^1} (p^2y^*, x)$, it must be that (px, y^*) is eliminated at the first stage. Thus, we conclude that $y^* \in E(\Gamma^2, \succsim)$.

Next, we show that for each $s \in S$, if $s \neq y^*$, then $s \notin E(\Gamma^2, \succsim)$.

Case 1: Suppose that $s = (px, y)$ for some $x, y \in X$ and $p \in (0,1)$. First, note that x can not be eliminated at the first stage, and for each $i \in \{1,2\}$, by *independence* of the preferences, we have $x \succ_i px$. Since for each $i \in \{1,2\}$, $\{i\} \in (px, y) \xrightarrow{\gamma^2} x$, it follows that (px, y) is eliminated at the second stage.

Case 2: Suppose that $s = y$ for some $y \in X \setminus \{y^*\}$. First, we show that there exists $i \in \{1,2\}$ such that $py^* \succ_i y$ for some $p \in (0,1)$. Suppose otherwise, then, by *continuity* of the preferences, for each $i \in \{1,2\}$, we have $y \succsim_i y^*$. Now, if $y \sim_1 y^*$ and $y \sim_2 y^*$, then by *independence*, we get that y is another Nash bargaining solution, contradicting the uniqueness of the solution. Therefore, it must be that $y \succ_i y^*$ for some $i \in \{1,2\}$. Then, by *continuity*, there exists $p \in (0,1)$ such that $py \succ_i y^*$. But, since y^* is the Nash bargaining solution, we must have for $j \neq i$, $py^* \succsim_j y$. Since, by *independence*, $y^* \succ_j py^*$, we obtain that $y^* \succ_j y$, which is a contradiction. Thus, we conclude that there exists $i \in \{1,2\}$ such that $py^* \succ_i y$ for some $p \in (0,1)$. Assume without loss of generality that $i = 1$.

Next, we show that (py^*, y) is not eliminated at the first stage. To see this, we show that $py^* \succ_1 p^2y$ and $py^* \succsim_2 p^2y$. By *independence*, we have $y \succ_1 p^2y$. Then, since $py^* \succ_1 y$, we have $py^* \succ_1 p^2y$. Now, since $y \neq y^*$and $py^* \succ_1 y$, by definition of the Nash solution, we must have $y^* \succsim_2 py$. Then, by *independence*, we have $py^* \succsim_2 p^2y$. Thus, we conclude that (py^*, y) is not eliminated at the first stage. Finally, since $\{1\} \in y \xrightarrow{\gamma^2} (py^*, y)$ and $py^* \succ_1 y$, we conclude that y is eliminated at the second stage. $\square$

5. Final comments

5.1 Relationship to Acemoglu, Egorov & Sonin (2012)

As a conceptual contribution, we extended the Koray & Yildiz (2018)'s notion of implementation via rights structures to multi-stages interactions. A related study is Acemoglu, Egorov & Sonin (2012) who propose a framework similar to (multi-stage) rights structures to study dynamic collective decision making problems. In this framework, a state determines players' payoffs as well as the coalitions that are entitled to move from the current state to the others.[2] In a two-stage rights structures, a state that arises as an equilibrium state at the first stage has a similar feature. In their analysis, authors focus on dynamic equilibria with sufficiently forward-looking agents, and characterize the set of dynamically stable states. In contrast, we are interested in designing multi-stage rights structures to implement given SCRs. Here, we assume that coalitions behave myopically in moving from one state to another.[3] We leave a comprehensive analysis of implementation via multi-stage rights structures, together with the case of farsighted behavior, as a future research agenda.

5.2 Relationship to classical implementation theory and Rubinstein (1995)

To demonstrate that two-stage rights structures can be fruitful in implementing SCRs, we show that the the *Nash bargaining solution* is implementable via a two-stage rights structures that has an intuitive interpretation. The implementation of Nash bargaining solution via game forms has been extensively studied in the literature. Among others, Binmore, Rubinstein & Wolinsky (1986) shows that a version of the alternating-offer

[2] A technical difference arises here, that is Acemoglu, Egorov & Sonin (2012) assume that if a coalition is entitled to move from one state, then the coalition is entitled to move to any arbitrary state. In a rights structures, the states from which a coalition is entitled to move can depend on the particular state that the coalition moves to.

[3] See Koray & Yildiz (2018)[Section 4.2] and Korpela, Lombardi & Vartiainen (2019) for implementation via (one-stage) rights structures with farsighted agents.

game form by Rubinstein (1982) approximately implements the Nash bargaining solution in subgame-perfect equilibrium. Howard (1992) formulates a four-stage extensive form game (with perfect information and chance moves) that exactly implements the Nash bargaining solution in subgame-perfect equilibrium.

A different approach that is closer to ours was proposed by Rubinstein (1995) who designs an *automaton* to obtain the subgame perfect equilibrium of the alternating-offer bargaining model. Each state s of the automaton specifies for each agent i an alternative set $A_i(s)$. The interpretation is that: if at state s agent i offers an alternative from $A_i(s)$, then this offer will be accepted by the other agent. As a key difference, the transition among the states happens depending on the *offers* and *accept/reject responses* of the agents who are assumed to be farsighted. Our view is that the notions of implementation via an automaton and ordinal-Nash solution, as Rubinstein (1995) puts it, "aim to shift the focus of theoretical bargaining models from *formulae* to *argumentation*." We view this study as a part of this general approach, since the two-stage rights structures constructed in Section 4 renders a similar, *argumentation* based, interpretation of the Nash bargaining solution.

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