On Capacity-Filling and Substitutable Choice Rules*

Battal Doğan† Serhat Doğan ‡ Kemal Yıldız§

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Abstract

Each *capacity-filling* and *substitutable* choice rule is known to have a maximizer-collecting representation: there exists a list of priority orderings such that from each choice set that includes more alternatives than the capacity, the choice is the union of the priority orderings' maximizers (Aizerman and Malishevski, 1981). We introduce the notion of a critical set and constructively prove that the number of critical sets for a choice rule determines its smallest size maximizer-collecting representation. We show that responsive choice rules require the maximal number of priority orderings in their smallest size maximizer-collecting representations among all *capacity-filling* and *substitutable* choice rules. We also analyze maximizer-collecting choice rules in which the number of priority orderings equals the capacity. We show that if the capacity is greater than three and the number of alternatives exceeds the capacity by at least two, then no *capacity-filling* and *substitutable* choice rule has a maximizer-collecting representation of the size equal to the capacity.

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†Department of Economics, University of Bristol; battal.dogan@bristol.ac.uk.

‡Department of Economics, Bilkent University; dserhat@bilkent.edu.tr.

§Department of Economics, Bilkent University; kemal.yildiz@bilkent.edu.tr.

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1 Introduction

Recent advances in market design have called for a better understanding of how institutions choose, or how they should choose, when faced with a set of alternatives. For example, in the context of assigning students to schools, it is important to understand the structure of plausible choice rules a school can use as an admissions policy. Although the relevant restrictions on choice rules vary across applications, *capacity-filling* and *substitutable* choice rules remain as the general prominent class. In this study, we provide a new representation result for *capacity-filling* and *substitutable* choice rules.

We consider a decision maker who has a capacity constraint and encounters choice problems each of which consists of a choice set. A choice rule, at each possible choice problem, chooses some alternatives from the choice set without exceeding the capacity. A choice rule is *capacity-filling* if it fills the capacity whenever possible, and accepts all alternatives from a choice set that includes no more alternatives than the capacity. *Capacity-filling* is a natural restriction in many applications where institutions prefer to fill their positions whenever possible.[1] A choice rule is *substitutable* if it chooses an alternative from a choice set whenever the alternative is chosen from a larger choice set.[2] That is, no alternative should be chosen because it complements another alternative. *Substitutability* has been a standard requirement in the market design literature following the seminal work of Kelso and Crawford (1982).[3] In the context of matching problems, Alkan (2001) presents the first model that studies

[1]In the matching literature, *capacity-filling* is also referred to as *acceptance*, although the *capacity-filling* terminology has been increasingly popular in the recent literature. Alkan (2001) is the first study which uses the "filling" terminology where he uses the term *quota filling*.

[2]In the revealed preference literature, *substitutability* is also referred to as *independence of irrelevant alternatives*, *Chernoff's condition* or *Sen's property* α.

[3]Beyond its normative appeal, Hatfield and Milgrom (2005) show that *substitutability* of choice rules guarantees the existence of a *stable* matching, which is a central desideratum for applications. Hatfield and Kojima (2008) show that *substitutability* of choice rules is an "almost necessary" condition for the non-emptiness of the core and the existence of *stable* allocations. Similarly, several classical results of matching literature have been generalized with *substitutable* choice rules (Roth and Sotomayor (1990), Alkan and Gale (2003), Hatfield and Milgrom (2005)).

substitutable choice rules together with *capacity-filling*.

Given capacity q, we say that a choice rule has a maximizer-collecting (MC) representation of size m, or simply called m-maximizer-collecting, if there exists a list of m priority orderings[4] such that all alternatives are chosen from each choice set that contains at most q alternatives, and the choice from each choice set that contains more than q alternatives is obtainable by collecting the maximizers of the priority orderings. It follows from Aizerman and Malishevski (1981) that each *capacity-filling* and *substitutable* choice rule has an MC representation.[5] However, the size of a smallest size MC representation of a choice rule and how to construct such a representation have been unknown.

We introduce the concept of a "critical set", which turns out to determine the minimal number of priority orderings required for an MC representation of a *capacity-filling* and *substitutable* choice rule. Given a choice rule, a choice set is a *critical set* if the number of alternatives in the choice set is equal to the capacity, and there exists an alternative that is chosen whenever added to the choice set, but no longer chosen whenever any other alternative is added afterwards (see Section 1 for a formal definition). In Theorem 1, we constructively prove that for each *capacity-filling* and *substitutable* choice rule, the number of priority orderings in its smallest size MC representation is equal to the number of its critical sets.

Well-known examples of *capacity-filling* and *substitutable* choice rules include responsive choice rules which have been studied particularly in the two-sided matching context (Gale and Shapley, 1962). A choice rule is responsive if there exists a priority ordering such that the choice from each choice set is obtainable by choosing the highest priority alternatives until the capacity is reached or no alternative is left. In Proposition 2, we show that the upper bound on the number of critical sets is achieved by responsive choice rules. That is, the size of the smallest size MC repre-

[4]A priority ordering is a complete, transitive, and anti-symmetric binary relation over all possible alternatives.

[5]Aizerman and Malishevski (1981) provide their representation result for *path independent* choice rules (Plott, 1973). It is known that *capacity-filling* and *substitutability* imply *path independence* (see Footnote 9 for more details).

sentation rendered by responsive choice rules is largest among all *capacity-filling* and *substitutable* choice rules.

We also analyze choice rules with MC representations of the size equal to the capacity q. For applications, q-MC choice rules have a particular appeal. If a choice rule is q-MC, then each one of the q priority orderings in its MC representation can be interpreted as a separate admission criterion for each available position. Put differently, the admission criterion for each position is represented by the associated priority ordering in the MC representation of size q. However, in Theorem 3, we show that if the capacity is greater than three and the number of alternatives exceeds the capacity by at least two, then no *capacity-filling* and *substitutable* choice rule has an MC representation of size q.

1.1 Related Literature

Capacity-filling together with *substitutability* imply *path independence* (Plott, 1973), which requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set. Among others, Plott (1973), Moulin (1985), Johnson (1990), and Johnson (1995) study the structure of *path independent* choice rules. Johnson and Dean (2001) and Koshevoy (1999) provide a lattice theoretic characterization of *path independent* choice rules.[6] Chambers and Yenmez (2017) use the MC representation of path independent choice rules to provide a new proof of a classical existence result for stable matching and a new result on welfare effects of expanding the choice rules in the matching context. Kojima and Manea (2010), Ehlers and Klaus (2014), and Ehlers and Klaus (2016) characterize deferred acceptance mechanisms where each school has a choice rule that satisfies *capacity-filling* and *substitutability*. Although the structure of *capacity-filling* and *substitutable* choice rules and their relation to matching mechanisms have been extensively studied, there is no direct implication of these studies for our con-

[6]They show that choice lattices associated with these rules constitute the class of lower locally distributive lattices.

struction of priority orderings that render a smallest size MC representation.

Cherepanov et al. (2013) analyze a two-stage choice model, in which an agent first shortlists alternatives that are optimal according to at least one of his *rationales*, and then singles out an alternative by maximizing a *preference order*.[7] They analyze choice functions that single out an alternative from each choice set, where a "rationale" is any asymmetric binary relation. In contrast, we consider choice rules that can be represented by collecting alternatives that are optimal according to a priority ordering, which is a complete, transitive, and antisymmetric binary relation. They put additional structure on the their representations by focusing on the maximality of the shortlisted alternatives,[8] which implicitly requires a possible increase in the number of used rationales. Here, we are interested in obtaining a minimal size representation, which goes in a sense to the opposite direction. Therefore, there is no direct implication of their results for our analysis.

As for our analysis of q-MC choice rules, the closest study is by Eliaz et al. (2011), who axiomatically analyze a decision maker who has in mind two orderings and chooses one or two alternatives that are maximizers of these orderings. This procedure, called *top-and-the-top*, yields a distinct pair of alternatives from each choice set, only if the second ordering is the inverse of the first ordering. In contrast, for any given capacity q, a q-MC choice rule chooses a set of q alternatives from each choice set that contains at least q alternatives.

2 Preliminaries

Let A be a nonempty finite set of n alternatives and let $\mathcal{A}$ denote the set of all nonempty subsets of A. A **choice rule** $C : \mathcal{A} \to \mathcal{A}$ associates with each choice set $S \in \mathcal{A}$, a nonempty set of alternatives $C(S) \subseteq S$. Let $q \in \{1, \ldots, n\}$ be a given capacity. We analyze choice rules that have the following two properties, which are well-known in the literature.

[7]This specified model is called "order rationalization".

[8]This is called "the minimum constraint theory of rationalization".

Capacity-filling: For given capacity $q \in \{1, \ldots, n\}$, an alternative is rejected from a choice set at a capacity q only if the capacity is full. Formally, for each $S \in \mathcal{A}$,

$$|C(S)| = \min\{|S|, q\}.$$

Substitutability: If an alternative is chosen from a choice set, then it is also chosen from any subset of the choice set that contains the alternative. Formally, for each $S, T \in \mathcal{A}$ such that $T \subseteq S$ and $a \in T$,

$$\text{if } a \in C(S), \text{ then } a \in C(T).$$

Each *capacity-filling* choice rule C satisfies *substitutability* if and only if C satisfies *path independence*: for each $S, S' \in \mathcal{A}$, $C(S \cup S') = C(C(S) \cup C(S'))$.[9] In plain words, *path independence* requires that if the choice set is "split up" into smaller sets, and if the choices from the smaller sets are collected and a choice is made from the collection, the final result should be the same as the choice from the original choice set (Plott, 1973).

The fact that *capacity-filling* together with *substitutability* imply *path-independence* follows from two facts. The first fact is that a choice rule is *path-independent* if and only if it is *substitutable* and satisfies *irrelevance of rejected alternatives* (*IRA*) (Aizerman and Malishevski, 1981). *IRA* requires that for all $S, S' \in \mathcal{A}$, $C(S) \subseteq S' \subseteq S$ implies $C(S) = C(S')$. The second fact is that *capacity-filling* together with *substitutability* imply *IRA*. Given $S, S' \in \mathcal{A}$ such that $C(S) \subseteq S' \subseteq S$, *substitutability* implies that $C(S) \subseteq C(S')$ and *capacity-filling* implies that $|C(S)| \geq |C(S')|$, which together imply that $C(S) = C(S')$.

Aizerman and Malishevski (1981) show that a choice rule is *path independent* if and only if there exists a list of priority orderings such that the choice from each choice set is the union of the highest priority alternatives in the priority orderings.[10] Next, we formally define and add more structure on these choice rules that we call MC choice rules.

[9] This equivalence is also noted in Remark 1 of Doğan and Klaus (2018), and it follows from Lemma 1 of Ehlers and Klaus (2016) together with Corollary 2 of Aizerman and Malishevski (1981).

[10] In the words of Aizerman and Malishevski (1981), each *path independent* choice rule is generable by some mechanism of collected extremal choice.

A **priority ordering** $\succ$ is a complete, transitive, and anti-symmetric binary relation over A. A **priority profile** $\pi = (\succ_1, \ldots, \succ_m)$, for some $m \in \mathbb{N}$, is an ordered list of m **distinct** priority orderings. Let Π denote the set of all priority profiles. Given $S \in \mathcal{A}$ and a priority ordering $\succ$, let $\max(S, \succ) = \{a \in S : \forall b \in S \setminus \{a\}, a \succ b\}$.

A choice rule C has a **maximizer-collecting (MC) representation** of size $m \in \mathbb{N}$ (or simply m**-maximizer-collecting**) if there exists $(\succ_1, \ldots, \succ_m) \in \Pi$ such that for each $S \in \mathcal{A}$ with $|S| \leq q$, $C(S) = S$, and for each $S \in \mathcal{A}$ with $|S| > q$, $C(S)$ is obtained by collecting the maximizers of the priority orderings in S, that is,

$$C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i).$$[11]

Next, we give examples of well-known *capacity-filling* and *substitutable* choice rules used in school choice applications.

Example 1 A choice rule C is **responsive** if there exists a priority ordering $\succ$ such that for each $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ$-priority alternatives until the capacity q is reached or no alternative is left.[12] Responsive choice rules have been studied particularly in the two-sided matching context (Gale and Shapley, 1962). The school choice literature, starting with the seminal study by Abdulkadiroğlu and Sönmez (2003), has widely focused on problems where each school is endowed with a responsive choice rule.

Example 2 A choice rule C is **lexicographic** if there exists a list of priority orderings $(\succ_1, \ldots, \succ_q) \in \Pi$ such that for each choice set $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ_1$-priority alternative in S, then choosing the highest $\succ_2$-priority alternative among the remaining alternatives, and so on until q alternatives are chosen or no alternative is left. As argued in detail by Kominers and Sönmez (2016), lexicographic choice rules have been particularly useful in designing allocation mechanisms for

[11]An alternative representation would require that for each $S \in \mathcal{A}$ with $|S| \leq q$, $C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i)$. See Section 3 Remark 1 for a discussion on such an alternative representation.

[12]Chambers and Yenmez (2018) show that a choice rule satisfies *capacity-filling* and the *weakened weak axiom of revealed priority (WWARP)* if and only if it is responsive.

school choice to achieve diversity.[13]

Example 3 Echenique and Yenmez (2015) consider a school choice problem in which students are partitioned into different types, each type t has its reserved seats, and there is a common priority ordering $\succ$, such as exam scores, that ranks all the students. A choice rule is **generated by reserves for priority** $\succ$ if for each type t, the highest priority students among type-t students are chosen until the reserves for type t are filled, or type-t students are exhausted. Then for the remaining seats, $\succ$-best students are chosen until all the seats or students are exhausted, that is students of all types compete against each other for all the seats that are not filled in the first stage. It follows from the characterization of Echenique and Yenmez (2015) that these choice rules satisfy *capacity-filling* and *substitutability*.

3 The representation result

Although it follows from Aizerman and Malishevski (1981) that a *capacity-filling* choice rule C satisfies *substitutability* if and only if C is MC, they remain silent about the smallest size MC representation and construction of the priority profile. In this section, we construct a smallest size MC representation of *capacity-filling* and *substitutable* choice rules.

We introduce the concept of a critical set for a choice rule, which will be the key in finding the minimal number of priorities needed for an MC representation.

Definition 1 *A choice set* $T \in \mathcal{A}$ *is a* ***critical set*** *for* C *if* $|T| = q$*, and there exists an alternative* $a \notin T$ *such that* $a \in C(T \cup \{a\})$ *and for each* $b \notin T \cup \{a\}$*,* $a \notin C(T \cup \{a, b\})$.

In plain words, given a choice rule C, a choice set is a critical set for C if the number of alternatives in the choice set is equal to the capacity and there exists an alternative not belonging to the choice set that is chosen whenever added to the choice set, but no longer chosen whenever any other alternative is added afterwards.

[13] See Doğan et al. (2017) for an axiomatic characterization of lexicographic choice rules in the school choice context.

The first result shows that given a *capacity-filling* and *substitutable* choice rule, the number of its critical sets determines the smallest size MC representation of the choice rule.

Theorem 1 *For each capacity-filling and substitutable choice rule C,*

i. C has an MC representation of a size equal to the number of its critical sets.

ii. C does not have an MC representation of any size smaller than the number of its critical sets.

First, we provide a sketch of the proof. The proof uses a specific choice lattice induced by a given *capacity-filling* and *substitutable* choice rule C.[14] The choice lattice is denoted by $(\mathcal{M}, \searrow)$, where $\mathcal{M}$ stands for, what we call, the set of *maximal choice sets* and $\searrow$ stands for a binary relation that we call the ancestor relation. A choice set is maximal if there is no larger choice set in which the same set of alternatives is chosen. A maximal choice set S is a parent of another choice set S' if $S' = S \setminus \{a\}$ for some $a \in C(S)$. A maximal choice set S is an ancestor of another choice set S' if there is a path that connects S to S' through parental relations. The collection of maximal choice sets endowed with *the ancestor relation* forms our choice lattice associated with C. For a pictorial representation of $(\mathcal{M}, \searrow)$ see the example in Section 4. In Lemmas 1-3, we prove some elementary but useful properties of $(\mathcal{M}, \searrow)$.

We observe that we can associate a priority ordering with each path in $(\mathcal{M}, \searrow)$ that connects A to a maximal choice set of cardinality q. It can be shown that the collection of all such possible priority orderings provides an MC representation of C, which in turn provides a proof of the Aizerman and Malishevski (1981) representation result. However, this obtained representation is not necessarily a minimal size representation.

The main idea behind the proof is to identify which paths are essential for an MC representation and how to use them to construct the representation. We discover

[14]It was first noted by Johnson (1990) that each *path independent* choice rule induces this specific choice lattice. Alkan (2001) and Chambers and Yenmez (2017) use a similar choice lattice in their proofs. We are grateful to Ahmet Alkan for bringing this to our attention.

that some maximal choice sets that we call "primes" and a collection of paths that spans all the primes are essential for representing the choice rule. That is, if we pick a collection of paths in $(\mathcal{M}, \searrow)$ that spans the primes (that is, for each prime, there is a path that contains the prime), then the associated priority profile represents the choice rule (see Lemma 7). The rest is to identify a minimal collection of paths that spans the primes. It turns out that if we collect each path connecting A to a prime with q alternatives, which we call a prime atom, then this set of paths spans all the primes.[15] Then, by using Lemma 7, we show that the associated priority profile renders an MC representation of the given choice rule.[16] Since each ordering is associated with a single prime atom, the number of orderings equals the number of prime atoms. Finally, we show that the number of priority orderings to represent C is at least the number of prime atoms. The fact that prime atoms coincide with critical sets (which directly follows from Lemma 4) completes the proof.

Proof of Theorem 1. Part i. Let C be a *capacity-filling* and *substitutable* choice rule. We first make some definitions, then introduce some lemmas, and finally prove the representation result. Note that C satisfies path independence as well (See Section 2).

A choice set is "maximal" if there is no larger choice set in which the same set of alternatives is chosen. Formally, a choice set $S \in \mathcal{A}$ is **maximal** for C if there is no larger set $T \supsetneq S$ with $C(S) = C(T)$. Let $\mathcal{M}$ denote the set of maximal choice sets for C.

We define the following binary relations on $\mathcal{M}$. For each $S, S' \in \mathcal{M}$, S is a **parent** of S', denoted by $S \to S'$, if there exists $a \in C(S)$ such that $S' = S \setminus \{a\}$.

[15] This conclusion does not hold with arbitrary path independent choice rules. We present an example in Footnote 18.

[16] There are two degree of freedoms in choosing these representations. One is that the last q alternatives can be ranked in any way. The other is the following. Suppose there is a prime node S which does not have a prime parent (if there is no prime, then the representation is unique up to the ranking of the last q alternatives). Now, there is exactly one priority in the representation that corresponds to the path containing S. The part of this path which connects S to a prime atom is unique (by Lemma 6). However the remaining part of the path that connects S to the grand set A can be chosen in any arbitrary way. This is the second degree of freedom in the construction.

There is a **path** from S to S' if there exists a collection of sets $S_1, \ldots, S_k \in \mathcal{M}$ such that $S \to S_1 \to \cdots \to S_k \to S'$. For each $S, S' \in \mathcal{M}$, S is an **ancestor** of S', denoted by $S \searrow S'$, if there is a path from S to S'. Note that the binary relation $\searrow$ is transitive and $(\mathcal{M}, \searrow)$ is a partially ordered set.

We call each $T \in \mathcal{M}$ such that $|T| = q$ as an **atom** of C.[17] A choice set $S \in \mathcal{M}$ with $|S| \geq q$ is a **prime** of C if S has a unique parent, that is, there exists a unique $S' \in \mathcal{M}$ such that $S' \to S$. Let $\mathcal{P}$ denote the set of all primes of C. A list of primes $(S_1, \ldots, S_k)$ such that $S_1 \to \cdots \to S_k$ is called a **prime path** from S_1 to S_k. An atom of C that is also a prime of C is called a **prime atom** of C. Let $\mathcal{P}_{\mathcal{A}}$ denote the set of all prime atoms of C.

Lemma 1 *For each choice set $S \in \mathcal{A}$, there exists a unique set $S' \in \mathcal{M}$ such that $C(S') = C(S)$.*

Proof. For each $S \in \mathcal{A}$, let $S' = \bigcup\{S_0 : C(S_0) = C(S)\}$. For each $S_1, S_2 \in \mathcal{A}$, if $C(S_1) = C(S_2) = C(S)$, then by *path independence*, $C(S_1 \cup S_2) = C(C(S_1) \cup C(S_2))$, which implies $C(S_1 \cup S_2) = C(S)$. It follows that S' is the unique maximal set with $C(S') = C(S)$ ■

Lemma 2 *For each choice set $S \in \mathcal{M}$ and each $a \in C(S)$, we have $S \setminus \{a\} \in \mathcal{M}$.*

Proof. By contradiction suppose there exist $S \in \mathcal{M}$ and $a \in C(S)$ such that $S \setminus \{a\} \notin \mathcal{M}$. Let $S' \in \mathcal{M}$ such that $C(S') = C(S \setminus \{a\})$. Since $S \setminus \{a\} \notin \mathcal{M}$, $S \setminus \{a\} \subsetneq S'$. Now, consider the choice set $S' \cup \{a\}$. Note that $S \subsetneq S' \cup \{a\}$. Moreover, since C is *path independent*, $C(S' \cup \{a\}) = C(C(S') \cup \{a\})$. Since $C(S') = C(S \setminus \{a\})$, we get $C(S' \cup \{a\}) = C(C(S \setminus \{a\}) \cup \{a\})$. Again by *path independence*, $C(C(S \setminus \{a\}) \cup \{a\}) = C(S)$. Thus, we get $C(S' \cup \{a\}) = C(S)$. Since $S \subsetneq S' \cup \{a\}$, this contradicts that $S \in \mathcal{M}$. ■

Lemma 3 *If a maximal choice set is a proper subset of another, then there is a path*

[17] An atom in a partially ordered set is an alternative that is minimal among all alternatives that are unequal to the least alternative. Since for constructing the desired priority profile, the choice sets with less than q alternatives do not play any role, we consider these choice sets as the least alternatives of $(\mathcal{M}, \searrow)$. Therefore, we call each choice set $T \in \mathcal{M}$ with q alternatives an atom of $(\mathcal{M}, \searrow)$.

from the larger set to the smaller. That is, for each $S, S' \in \mathcal{M}$ such that $S' \subsetneq S$, we have $S \searrow S'$.

Proof. Let $S, S' \in \mathcal{M}$ be such that $S' \subsetneq S$. Since $S, S' \in \mathcal{M}$ and C is *path independent*, there exists $a \in C(S) \setminus S'$. Suppose otherwise, i.e. suppose that $C(S) \subset S'$. It follows from *path independence* that $C(C(S) \cup S') = C(C(S) \cup C(S')) = C(S \cup S') = C(S)$. Since $C(S) \cup S' = S'$, we get $C(S) = C(S')$, which contradicts that $S' \in \mathcal{M}$. Thus, there exists $a \in C(S) \setminus S'$. Now, let $S_1 = S \setminus \{a\}$. Since $S \in \mathcal{M}$, it follows from Lemma 2 that $S_1 \in \mathcal{M}$. Thus, we get $S \to S_1$. To get S_2, note that since $S' \subset S_1$, there exists $b \in C(S_1) \setminus S'$. By proceeding similarly we obtain a path $(S_1, \ldots, S_k)$ from S to S'. ■

Lemma 4 *Suppose that $S \in \mathcal{M}$ with $|S| \geq q$, and $S \cup \{a\}$ is a parent of S. Then, S is a prime if and only if a is no longer chosen whenever any other alternative is added to $S \cup \{a\}$, that is for each $b \notin S \cup \{a\}$, $a \notin C(S \cup \{a, b\})$.*

Proof. (*If part*) We prove the contrapositive statement. Assume that $S \in \mathcal{M}$ is not a prime. Then, there exists $b \notin S \cup \{a\}$ such that $S \cup \{b\}$ is a parent of S. Since $S \cup \{a\}$ is a parent of S, by definition, we have $S \cup \{a\} \in \mathcal{M}$. Suppose that $a \notin C(S \cup \{a, b\})$. Then, since C satisfies *substitutability* and *capacity-filling* and $S \cup \{a\} \subset S \cup \{a, b\}$, it follows that $C(S \cup \{a, b\}) = C(S \cup \{b\})$. But, this contradicts $S \cup \{b\} \in \mathcal{M}$. Hence $a \in C(S \cup \{a, b\})$.

(*Only if part*) We prove the contrapositive statement. Suppose that there exists $b \notin S \cup \{a\}$ such that $a \in C(S \cup \{a, b\})$. By Lemma 1, there exists $S' \cup \{a, b\} \in \mathcal{M}$ such that $C(S \cup \{a, b\}) = C(S' \cup \{a, b\})$, and $a, b \notin S'$. Note that by the choice of S', $S \subset S'$. Since $a \in C(S' \cup \{a, b\})$, by Lemma 2, $S' \cup \{b\} \in \mathcal{M}$. Then, since we also have $S \in \mathcal{M}$, it follows from Lemma 3 that $S' \cup \{b\} \searrow S$. Since $a \notin S' \cup \{b\}$, the path connecting $S' \cup \{b\}$ to S must reach S via a parent other than $S \cup \{a\}$. Therefore, S is not a prime. ■

Lemma 5 *Each prime that is not an atom is the parent of a unique prime. That is, for each $S \in \mathcal{P}$ such that $|S| > q$, there exists a unique $S' \in \mathcal{P}$ such that $S \to S'$.*[18]

[18] As we emphasize before, this result fails if the choice rule C is *path independent* but not *capacity-filling*. To see this by an example, let $A = \{1, 2, 3, 4\}$, and consider the following two priorities:

Proof. Let $S \in \mathcal{M}$ with $|S| > q$. Suppose that S is a prime. Let $S \cup \{b^*\}$ be the unique parent of S. Since $b^* \in C(S \cup \{b^*\})$ and C is *capacity-filling*, there exists $a \in C(S) \setminus C(S \cup \{b^*\})$. Consider the choice set $S \setminus \{a\}$. Since $S \in \mathcal{M}$ with $|S| > q$, and $a \in C(S)$, it follows from Lemma 2 that $S \setminus \{a\} \in \mathcal{M}$. It also follows that $S \to S \setminus \{a\}$.

Next, we show that $S \setminus \{a\}$ is a prime. By contradiction, suppose $S \setminus \{a\}$ is not a prime. Lemma 4 implies that there exists $b \notin S$ such that $a \in C(S \cup \{b\})$. Since $a \notin C(S \cup \{b^*\})$, $b \neq b^*$. Next, consider the choice set $S \cup \{b^*, b\}$. Since S is a prime, $S \cup \{b^*\} \to S$, and $b \notin S \cup \{b^*\}$, it follows from Lemma 4 that $b^* \notin C(S \cup \{b^*, b\})$. By *path independence*, we have $C(S \cup \{b^*, b\}) = C(C(S \cup \{b\}) \cup \{b^*\})$. Since $b^* \notin C(S \cup \{b^*, b\})$, we get $C(S \cup \{b^*, b\}) = C(S \cup \{b\})$. Now, since $a \in C(S \cup \{b\})$, we have $a \in C(S \cup \{b^*, b\})$. But we also have $a \notin C(S \cup \{b^*\})$, contradicting that C satisfies *substitutability*. Thus, we obtain that $S \setminus \{a\}$ is a prime. Since we know that $S \to S \setminus \{a\}$, $S \setminus \{a\}$ is a prime child of S.

To see that the prime child is unique, by contradiction, suppose that there exist distinct $a, a' \in C(S)$ such that $S \to S \setminus \{a\}$ and $S \to S \setminus \{a'\}$, where both $S \setminus \{a\}$ and $S \setminus \{a'\}$ are prime. Now, since A is not a prime, $S \neq A$, and there exists some $x \notin S$. Next, consider $S \cup \{x\}$. It follows from Lemma 4 that $a, a' \notin C(S \cup \{x\})$. This combined with C being *capacity-filling* implies there exists $y \in C(S \cup \{x\}) \setminus C(S)$ such that $y \neq x$. This directly contradicts that C satisfies *substitutability*. ■

Lemma 6 *For each prime $S \in \mathcal{P}$ that is not an atom, there exists a unique prime path from S to a prime atom $T \in \mathcal{P}$. Moreover, the unique prime path from S to T is included in any path from A to T. Formally, for each $S \in \mathcal{P}$ such that $|S| > q$, there exists a unique list of primes $(S_1, \dots, S_k)$ such that $S \to S_1 \cdots \to S_k \to T$, where T is a prime atom and for each $t \in \{1, \dots, k\}$, S_t is included in each path from A to T.*

$1 \succ 2 \succ 3 \succ 4$ and $1 \succ' 4 \succ' 3 \succ' 2$. Let C be the maximizer collecting of $\succ$ and $\succ'$. It directly follows that C is *path independent*. But, since $C(A) = \{1\}$ and $C(A \setminus \{1\}) = \{2, 4\}$, C is not *capacity-filling*. To see that C fails to satisfy the conclusion of the Lemma, note that $\{2,3,4\}, \{2,3\}, \{3,4\}$ are prime choice sets. However, since both $\{2,3,4\} \to \{3,4\}$ and $\{2,3,4\} \to \{2,3\}$, $\{2,3,4\}$ is the parent of two primes. Note that A having four alternatives does not play any essential role in this argument, so we can construct similar examples for any universal set of alternatives.

Proof. By Lemma 5, there exists a unique prime $S_1 \in \mathcal{P}$ such that $S \to S_1$. Applying Lemma 5 consecutively, there exists a unique list of primes $(S_1, \ldots, S_k)$ such that $S \to S_1 \cdots \to S_k \to T$, where T is a prime atom. Now, since for each $t \in \{1, \ldots, k\}$, S_t is prime and T is a prime atom, each S_t and T have a unique parent. It follows that for each $t \in \{1, \ldots, k\}$, S_t must be included in any path that connects A to T. ■

Lemma 7 *Suppose that S is a nonprime maximal choice set with $S \neq A$, and $S \cup \{a\}$ is its parent. There exists a maximal choice set $S' \in \mathcal{M}$ such that $S \cup \{a\} \subsetneq S'$ and $a \in C(S')$.*

Proof. Since $S \neq A$ and S is not a prime, there exists $b \notin S \cup \{a\}$ such that $S \cup \{b\} \to S$. Thus, $b \in C(S \cup \{b\})$. Now, consider the choice set $S \cup \{a, b\}$. We show that $a \in C(S \cup \{a, b\})$. Suppose $a \notin C(S \cup \{a, b\})$, then since C satisfies *path independence*, $C(S \cup \{a, b\}) = C(S \cup \{b\})$, which is a contradiction since $S \cup \{b\} \in \mathcal{M}$. Hence, $a \in C(S \cup \{a, b\})$. Now, by Lemma 1, there exists $S' \in \mathcal{M}$ such that $C(S') = C(S \cup \{a, b\})$. Since S' is maximal, $S \cup \{a, b\} \subset S'$, which implies $S \cup \{a\} \subsetneq S'$, and $a \in C(S')$. Therefore, S' is the desired choice set. ■

Now, we are ready to construct the set of priority orderings for the desired representation. For each prime atom T, we pick a path from A to T. Note that there must be at least one such path since A is the maximal alternative in the choice lattice, and, by Lemma 3, there is a path from A to any other $S \in \mathcal{M}$ with $|S| \geq q$. We construct a priority ordering using such a path as follows. Consider a prime atom $T \in \mathcal{P}_{\mathcal{A}}$. Take one path from A to T, say, $S_1, \ldots, S_k \in \mathcal{M}$ such that $A \to S_1 \to \cdots \to S_k \to T$. Let $\{a_1\} = A \setminus S_1$, $\{a_{k+1}\} = S_k \setminus T$, and for each $i \in \{2, \ldots, k\}$, $\{a_i\} = S_{i-1} \setminus S_i$. Note that by definition of a parent, for each $i \in \{1, \ldots, k+1\}$, a_i is well-defined. Now, let $\succ^T$ be such that for each $i, j \in \{1, \ldots, k+1\}$, $a_i \succ^T a_j$ if $i < j$, and assume that any other remaining alternative is ranked below a_{k+1} arbitrarily. Thus, we obtain a set of priority orderings $(\succ^T)_{T \in \mathcal{P}_{\mathcal{A}}}$ by constructing a priority ordering for each prime atom.

Note that the number of priority orderings in $(\succ^T)_{T \in \mathcal{P}_{\mathcal{A}}}$ is equal to the number of prime atoms. Moreover, we claim that there are at least q priority orderings, or in other words there are at least q prime atoms. To see this, first note that each maximal set $S \in \mathcal{M}$ such that $A \to S$ is a prime. Also, since $|C(A)| = q$, there are q

such primes. Suppose that the number of prime atoms is less than q. Note that there are q prime choice sets $\{A \setminus \{a_i\}\}_{a_i \in C(A)}$ that are children of A. Since by Lemma 6, there is a prime path from any prime to a prime atom, supposing that the number of prime atoms is less than q implies there exist two different primes S and S' that are children of A such that prime paths originating from S and S' connect them to the same prime atom T. It follows that these two prime paths originating from S and S' merge at some choice set. This implies that the first (largest) choice set contained by both of these prime paths has two different parents. But, this contradicts that this choice set is a prime.

In two steps, we show that for each $S \in \mathcal{A}$ such that $|S| > q$,

$$C(S) = \bigcup_{T \in \mathcal{P}_{\mathcal{A}}} \max(S, \succ^T). \tag{1}$$

First, we show that if (1) holds for each maximal set, then it holds for each choice set. To see this, suppose that (1) holds for each maximal set. Take any $S \notin \mathcal{M}$ such that $|S| > q$. Then, by Lemma 1, there exists $S' \in \mathcal{M}$ such that $S \subset S'$ and $C(S) = C(S')$. Since (1) holds for S', $C(S') = \bigcup_{T \in \mathcal{P}_{\mathcal{A}}} \max(S', \succ^T)$. But since $S \subseteq S'$ and $C(S) = C(S')$, no alternative in $S' \setminus S$ belongs to $C(S')$. Thus, there is no $a \in S' \setminus S$ and $T \in \mathcal{P}_{\mathcal{A}}$ such that $a = \max(S', \succ^T)$. But then, $\bigcup_{T \in \mathcal{P}_{\mathcal{A}}} \max(S', \succ^T) = \bigcup_{T \in \mathcal{P}_{\mathcal{A}}} \max(S, \succ^T)$. Hence, $C(S) = \bigcup_{T \in \mathcal{P}_{\mathcal{A}}} \max(S, \succ^T)$.

Second, we show that (1) holds for each maximal set. Let $S \in \mathcal{M}$ be such that $|S| > q$. First, we show that $\bigcup_{T \in \mathcal{P}_{\mathcal{A}}} \max(S, \succ^T) \subset C(S)$. Let $T \in \mathcal{P}_{\mathcal{A}}$ and let $S_1, \ldots, S_k \in \mathcal{M}$ be such that $A \to S_1 \to \cdots \to S_k \to T$ is the path from A to T used in the construction of $\succ^T$. Suppose $a = max(S, \succ^T)$. Then, by the construction of $\succ^T$, there exists $t \in \{0, 1, \ldots, k\}$ such that $\{a\} = S_t \setminus S_{t+1}$, where $S_0 = A$. By the construction of $\succ^T$, this means for each $b \in A \setminus S_t$, $b \succ^T a$. Since $a = max(S, \succ^T)$, it follows that $S \subseteq S_t$. To see that $a \in C(S_t)$, first, note that since C satisfies *path independence*, $C(S_{t+1} \cup \{a\}) = C(C(S_{t+1}) \cup \{a\})$. Since $\{a\} = S_t \setminus S_{t+1}$, we can have $S_t, S_{t+1} \in \mathcal{M}$ only if $a \in C(S_t)$. Now, since C satisfies *substitutability* and $S \subseteq S_t$, we get $a \in C(S)$.

Next, we show that for each maximal set $S \in \mathcal{M}$ with $|S| > q$, $C(S) \subset$

$\bigcup_{T\in\mathcal{P}_\mathcal{A}} \max(S,\succ^T)$. Suppose that $a \in C(S)$. If we can find a set $S^* \in \mathcal{M}$ with $a \in C(S^*)$, $S \subset S^*$ and $S^* \setminus \{a\} \in \mathcal{P}$, then by Lemma 6 we know there exist a unique $T \in \mathcal{P}_\mathcal{A}$ which can be reached via a unique prime path from $S^* \setminus \{a\}$. Moreover, again by Lemma 6, the unique prime path from $S^* \setminus \{a\}$ to T is included in any path from A to T. Now, note that $S^* \setminus \{a\}$ is a prime so any path including $S^* \setminus \{a\}$ must pass through S^*. Then, S^* is included in any path from A to T. Thus by construction of $\succ^T$ we will have $a = \max(S^*,\succ^T) = \max(S,\succ^T)$ and this will conclude the proof.

We conclude by showing that there exists $S^* \in \mathcal{M}$ such that $a \in C(S^*)$, $S \subset S^*$ and $S^* \setminus \{a\} \in \mathcal{P}$. Since $S \in \mathcal{M}$ and $a \in C(S)$, by Lemma 2, $S \setminus \{a\} \in \mathcal{M}$ as well. If $S \setminus \{a\} \in \mathcal{P}$, then set $S^* = S$ and we are done. Suppose that $S \setminus \{a\}$ is not a prime. By Lemma 7 there is $S' \in \mathcal{M}$ such that $S \subsetneq S'$ and $a \in C(S')$. If $S' \setminus \{a\}$ is prime, then set $S^* = S'$ and we are done. Otherwise, we continue in the same fashion. Since A is finite and, by definition of a prime, $A \setminus \{b\}$ is prime for any $b \in C(A)$, then after a finite number of steps we must eventually come across a set with the desired properties.

Part ii. Let C be a *capacity-filling* and *substitutable* choice rule. Suppose that C has an MC representation for the priority profile $(\succ_1, \ldots, \succ_m)$. Consider the set of prime atoms $\mathcal{P}_\mathcal{A}$ of C. In what follows we show that $m \geq |\mathcal{P}_\mathcal{A}|$.

For each $T \in \mathcal{P}_\mathcal{A}$, since T is prime, there exists a unique $a \notin T$ such that $T \cup \{a\} \to T$. Since $a \in C(T \cup \{a\})$, there exists $\succ_T \in \{\succ_1, \ldots, \succ_m\}$ such that $a = max(T \cup \{a\}, \succ_T)$. We will show that for each distinct $T, T' \in \mathcal{P}_\mathcal{A}$, $\succ_T \neq \succ_{T'}$.

First, we show that, $T = \{b \in A : a \succ_T b\}$. Since $a = max(T \cup \{a\}, \succ_T)$, $T \subset \{b \in A : a \succ_T b\}$. To see the converse, suppose that there exists $b \notin T$ such that $a \succ_T b$. Then $a = max(T \cup \{a, b\}, \succ_T)$, which implies that $a \in C(T \cup \{a, b\})$. This is a contradiction, since Lemma 4 implies that a is no longer chosen whenever any other alternative is added to $T \cup \{a\}$. Thus $\{b \in A : a \succ_T b\} \subset T$, which implies $T = \{b \in A : a \succ_T b\}$. Now, since $|T| = q$, T constitutes the bottom-ranked q alternatives at $\succ_T$. By similar arguments, T' constitutes the bottom-ranked q alternatives at $\succ_{T'}$. But then, $\succ_T \neq \succ_{T'}$ since $T \neq T'$. Hence, prime atoms are associated with distinct priority orderings, which implies that $m \geq |\mathcal{P}_{\mathcal{A}}|$. ■

Remark 1 First, we would like to note that it is not clear if a similar result can be obtained for all *path independent* choice rules, in that one of our key observations, namely Lemma 5, does not hold in the absence of *capacity-filling*.[19] Second, let us consider the following alternative representation: a choice rule C is maximizer-collecting* if there exists a priority profile $\pi = (\succ_1, \ldots, \succ_m)$ such that for each $S \in \mathcal{A}$, including the choice sets with $|S| \leq q$, $C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i)$. A choice set T is a critical* set if $|T| = q - 1$ and there exists $a \notin T$ such that a is chosen from $T \cup \{a\}$, but no longer chosen whenever any other alternative is added to $T \cup \{a\}$. The only difference between a critical set and a critical* set is that a critical set has q alternatives, whereas a critical* set has $q - 1$ alternatives. It turns out that Theorem 1 holds with maximizer-collecting* choice rules and critical* sets.[20] We prefer maximizer-collecting representations since under *capacity-filling*, asking for a maximizer-collecting* representation does not change the set of representable rules, but requires adding, in a sense "unnecessary", priorities to the representation.

3.1 The case of responsive choice rules

A well-known example of a *capacity-filling* choice rule that satisfies *substitutability* is a responsive choice rule (see Example 1 for the definition). In Theorem 2, we show that the upper bound on the number of critical sets is achieved by responsive choice rules. Put differently, the size of the smallest size MC representation rendered by responsive choice rules is largest among all *capacity-filling* and *substitutable* choice

[19]We presented a *path independent* choice rule that fails to satisfy Lemma 5 in Footnote 18.

[20]Our construction in the proof of Theorem 1 can be easily extended to obtain this result. To see this, consider the paths from the universal set of alternatives to prime atoms. As mentioned in the sketch of the proof, we obtain the priority orderings out of these paths. For each of these priority orderings, we are free to order the last $q - 1$ alternatives. For the extension, consider all the paths that connect the universal set of alternatives to prime* atoms. Similarly, the priority profile associated with these paths provides a smallest size maximizer-collecting* representation. For this more stringent representation, one should additionally be careful about the $(n-q)^{th}$-ranked alternative at each priority. These longer paths discipline which alternatives are ranked at the $(n-q)^{th}$ position.

rules.

Theorem 2 *For each capacity constraint q and universal set of n alternatives with $n \geq q+2$,*[21] *each capacity-filling and substitutable choice rule C has an MC representation of a size less than or equal to $\binom{n-2}{q-1}$. If C is responsive, then C does not have an MC representation of any size smaller than $\binom{n-2}{q-1}$.*

It is not difficult to see that the number of critical sets for a responsive choice rule is $\binom{n-2}{q-1}$. Let C be a *capacity-filling* choice rule that is responsive to the priority ordering $\succ$. Let $a \in A$ be the $(n-1)^{th}$-ranked and b be the $(n)^{th}$-ranked alternative at $\succ$. Clearly, a choice set $S \in \mathcal{A}$ such that $|S| = q$ is a critical set if and only if $a \notin S$ and $b \in S$. Since there are $\binom{n-2}{q-1}$ such sets, C has $\binom{n-2}{q-1}$ critical sets. The difficult part of the proof is to show that for an arbitrary *capacity-filling* and *substitutable* choice rule, the number of its critical sets is at most $\binom{n-2}{q-1}$. To show this, we prove a structural result, which states that for each *capacity-filling* and *substitutable* choice rule and for each $k \in \{q, \ldots, n\}$, there are exactly $\binom{n-k+q-1}{q-1}$ maximal choice sets with cardinality k (in particular, the number of maximal choice sets with a fixed cardinality is invariant among *capacity-filling* and *substitutable* choice rules). We formalize this structural result in Lemma 8. We denote the collection of maximal choice sets with cardinality k by $\mathcal{M}_k$.

Lemma 8 *Given a capacity q and a universal set of n alternatives, let C be a capacity-filling and substitutable choice rule. For each $k \in \{q, \ldots, n\}$, the number of maximal choice sets with cardinality k is $\binom{n-k+q-1}{q-1}$, i.e. $|\mathcal{M}_k| = \binom{n-k+q-1}{q-1}$.*

Proof. First, we argue that if for each $k \in \{q, \ldots, n\}$ the following identity holds, then we obtain the desired conclusion.

$$\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \sum_{i=k}^{n} \binom{i-q}{k-q} \binom{n-i+q-1}{q-1} \tag{2}$$

To see this, note that $|\mathcal{M}_n| = 1$ and for $k = n-1$, it follows from (2) that

$$\binom{n-1-q}{k-q} |\mathcal{M}_{n-1}| + \binom{n-q}{k-q} |\mathcal{M}_n| = \binom{n-1-q}{k-q} \binom{q}{q-1} + \binom{n-q}{k-q} \binom{q-1}{q-1}.$$

[21] For $n = q+1$, the smallest size MC representation is q. We show this after proving Theorem 3.

Since $|\mathcal{M}_n| = 1 = \binom{q-1}{q-1}$, we have $|\mathcal{M}_{n-1}| = \binom{q}{q-1}$. Similarly for $k = n-2$, we have $|\mathcal{M}_{n-2}| = \binom{q+1}{q-1}$. Proceeding inductively we obtain that $|\mathcal{M}_k| = \begin{pmatrix} n-k+q-1 \\ q-1 \end{pmatrix}$. In what follows we prove that (2) holds in two steps by showing that both sides of the equality are equal to $\binom{n}{k}$.

Step 1. We show that $\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \binom{n}{k}$. To see this, first, consider $\mathcal{K} = \{S \in \mathcal{A} : |S| = k\}$. Then, consider the partition of $\mathcal{K}$ such that for each $S, S' \in \mathcal{K}$, S and S' belong the same part if and only if $C(S) = C(S')$. First, we show that

$$\mathcal{K} = \bigcup_{i=k}^{n} \bigcup_{S' \in \mathcal{M}_i} \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\} \tag{3}$$

Since for each $S \in \mathcal{A}$, there exists a unique $S' \in \mathcal{M}$ such that $C(S) = C(S')$ and $S \subset S'$, we get

$$\mathcal{K} = \bigcup_{S' \in \mathcal{M}} \{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} \tag{4}$$

Since $\{\mathcal{M}_i\}_{i=k}^{n}$ partitions $\{S' \in \mathcal{M} : |S'| \geq k\}$, we can rewrite (4) as

$$\mathcal{K} = \bigcup_{i=k}^{n} \bigcup_{S' \in \mathcal{M}_i} \{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} \tag{5}$$

Finally, note that for each $S' \in \mathcal{M}$ and $S \in \mathcal{K}$, if $C(S') \subset S \subset S'$, then *substitutability* implies that $C(S') = C(S)$. Therefore, $\{S \in \mathcal{A} : |S| = k, C(S) = C(S')\} = \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}$. This observation together with (5) implies that (3) holds.

Now, if we count both sides of (3), then we obtain

$$\binom{n}{k} = \sum_{i=k}^{n} \sum_{S' \in \mathcal{M}_i} |\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}| \tag{6}$$

Next, we argue that for each $i \in \{k, \ldots, n\}$, and $S' \in \mathcal{M}_i$,

$$|\{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}| = \binom{i-q}{k-q} \tag{7}$$

To see this, for each $i \in \{k, \ldots, n\}$, and $S' \in \mathcal{M}_i$, consider the set $\{T \subset S' \setminus C(S') : |T| = k-q\}$. Since $S' \in \mathcal{M}_i$, $|S' \setminus C(S')| = i - q$. It directly follows that $|\{T \subset S' \setminus C(S') : |T| = k-q\}| = \binom{i-q}{k-q}$. To show that (7) holds, we argue that $F = \{S \in \mathcal{A} : |S| = k, C(S') \subset S \subset S'\}$ is isomorphic[22] to $F' = \{T \subset S' \setminus C(S') : |T| = k-q\}$. To

[22]That is, there is a bijection between the two sets.

see this we define the mapping g such that for each $S \in F$, $g(S) = S \setminus C(S')$. Since for each $S \in F$, $C(S') \subset S \subset S'$ and $|S| = k$, we have $|g(S)| = k - q$ and $g(S) \in F'$. Thus $g : F \to F'$. Since for each distinct $S_1, S_2 \in F$, $S_1 \setminus C(S') \neq S_2 \setminus C(S')$, g is one-to-one. Since for each $T \in F'$, $g(T \cup C(S')) = T$, g is onto. Therefore g is a bijection between F and F'. Thus, we obtain that (7) holds. Finally, if we combine (6) and (7), then we directly obtain that $\sum_{i=k}^{n} \binom{i-q}{k-q} |\mathcal{M}_i| = \binom{n}{k}$.

Step 2. We show that $\sum_{i=k}^{n} \binom{i-q}{k-q}\binom{n-i+q-1}{q-1} = \binom{n}{k}$. To see this consider the set $\{1, 2, \ldots, n\}$ and let $\mathcal{K}$ be the collection of its subsets that contain k alternatives. Since there are $\binom{n}{k}$ such subsets, $|\mathcal{K}| = \binom{n}{k}$. Since $k > q$, for each $S \in \mathcal{K}$, there exists $q(S) \in S$ that is the q^{th} highest number in S. Now, consider the partition of $\mathcal{K}$ such that for each $S, S' \in \mathcal{K}$, S and S' belong to the same part if and only if $q(S) = q(S')$. We denote this partition of $\mathcal{K}$ by $\mathcal{L}$. Now, note that for each $S \in \mathcal{K}$, $q(S) \in \{k+1-q, \ldots, n+1-q\}$. Next, for each $j \in \{k+1-q, \ldots, n+1-q\}$, we count the number of $S \in \mathcal{K}$ such that $q(S) = j$. If $q(S) = j$, then there are $k - q$ numbers in S that are less than j, and $q - 1$ numbers that are greater than j. It follows that S can be chosen in $\binom{j-1}{k-q}\binom{n-j}{q-1}$ different ways. This observation together with $\mathcal{L}$ partitions $\mathcal{K}$ implies that

$$\binom{n}{k} = \sum_{j=k+1-q}^{n+1-q} \binom{j-1}{k-q}\binom{n-j}{q-1} \tag{8}$$

A standard change of variables with $j = i + 1 - q$ here yields that the right-hand side of (8) equals $\sum_{i=k}^{n} \binom{i-q}{k-q}\binom{n-i+q-1}{q-1}$. Thus, we obtain the desired equality. ■

Now, we prove Theorem 2 using Theorem 1, Lemma 8, and Lemma 4.

Proof of Theorem 2. Let $n \geq q+2$ and C be a *capacity-filling* and *substitutable* choice rule. Consider the maximal choice sets with $q+1$ alternatives, i.e. $\mathcal{M}_{q+1} = \{S \in \mathcal{M} : |S| = q+1\}$. Note that since $n \geq q+2$, $A \notin \mathcal{M}_{q+1}$. Next, we show that each $S \in \mathcal{M}_{q+1}$ has at most one prime child. By contradiction, suppose there exist $S \in \mathcal{M}_{q+1}$ and distinct $a, b \in S$ such that $S \setminus \{a\}$ and $S \setminus \{b\}$ are prime children of S. Since $S \neq A$, let $x \notin S$ be such that $S \cup \{x\}$ is a parent of S. Now, since $S \cup \{x\}$ is a parent of S, $x \in C(S \cup \{x\})$. Since $|S \cup \{x\}| = q + 2$ and C is *capacity-filling*, $|C(S \cup \{x\})| = q$. Since C satisfies *substitutability*, it follows that $|C(S \cup \{x\}) \cap C(S)| = q - 1$. Since

$S \setminus \{a\}$ and $S \setminus \{b\}$ are prime children of S, by Lemma 4, we have $a \notin C(S \cup \{x\})$ and $b \notin C(S \cup \{x\})$, although $a, b \in C(S)$ and $|C(S) = q|$. This contradicts that $|C(S \cup \{x\}) \cap C(S)| = q - 1$. Therefore, the number of prime atoms is at most $|\mathcal{M}_{q+1}|$. By Lemma 8, $|\mathcal{M}_{q+1}| = \binom{n-2}{q-1}$. Thus, $\binom{n-2}{q-1}$ is an upper bound on the number of prime atoms. Hence, by Theorem 1, C has an MC representation of a size less than or equal to $\binom{n-2}{q-1}$.

Let C be a *capacity-filling* choice rule that is responsive to the priority ordering $\succ$. Let $a \in A$ be the $(n-1)^{th}$-ranked and $b \in A$ be the $(n)^{th}$-ranked alternative at $\succ$. Clearly, a set $S \in \mathcal{A}$ such that $|S| = q$ is a prime atom if and only if $a \notin S$ and $b \in S$. Since there are $\binom{n-2}{q-1}$ such sets, C has $\binom{n-2}{q-1}$ prime atoms. Then, by Theorem 1, C does not have an MC representation of any size smaller than $\binom{n-2}{q-1}$. $\blacksquare$

Responsive choice rules are not unique in having maximum number of critical sets. To show this, in the next example, we construct a choice rule that is not responsive, but the number of its critical sets equals $\binom{n-2}{q-1}$.

Example 4 Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider the priority profile $(\succ_\alpha, \succ_\beta, \succ_\gamma, \succ_\delta)$. Let the capacity be 2 and let C be the *capacity-filling* and *substitutable* choice rule that has an MC representation via the following priority profile.

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
1	1	2	2
3	2	3	3
4	4	4	1
5	5	5	6
6	6	6	5
2	3	1	4

The choice lattice $(\mathcal{M}, \searrow)$ associated with C is depicted in Figure 1.[23] It directly follows from the structure of this choice lattice and Lemma 4 that C has 4 critical sets, namely $\{1, 6\}, \{4, 5\}, \{3, 6\}, \{2, 6\}$, which is the upper bound on the number

[23]Maximal choice sets are denoted by black circles and prime choice sets are denoted by green circles (gray circles in B&W print).

of critical sets. Next, we argue that C is not responsive. Suppose otherwise, i.e., suppose that C is responsive to a priority ordering $\succ$. Since $C(\{2,4,5,6\}) = \{2,4\}$, 2 and 4 should be the two best $\succ$-ranked alternatives in this choice set. Let x be the next best ranked $\succ$-alternative in the same choice set. Now, we should have $C(\{2,5,6\}) \setminus \{2\} = C(\{4,5,6\}) \setminus \{4\} = x$. However, this is not the case, since when 4 is removed from $\{4,5,6\}$, 5 is chosen; but when 2 is removed from $\{2,5,6\}$, 6 is chosen.

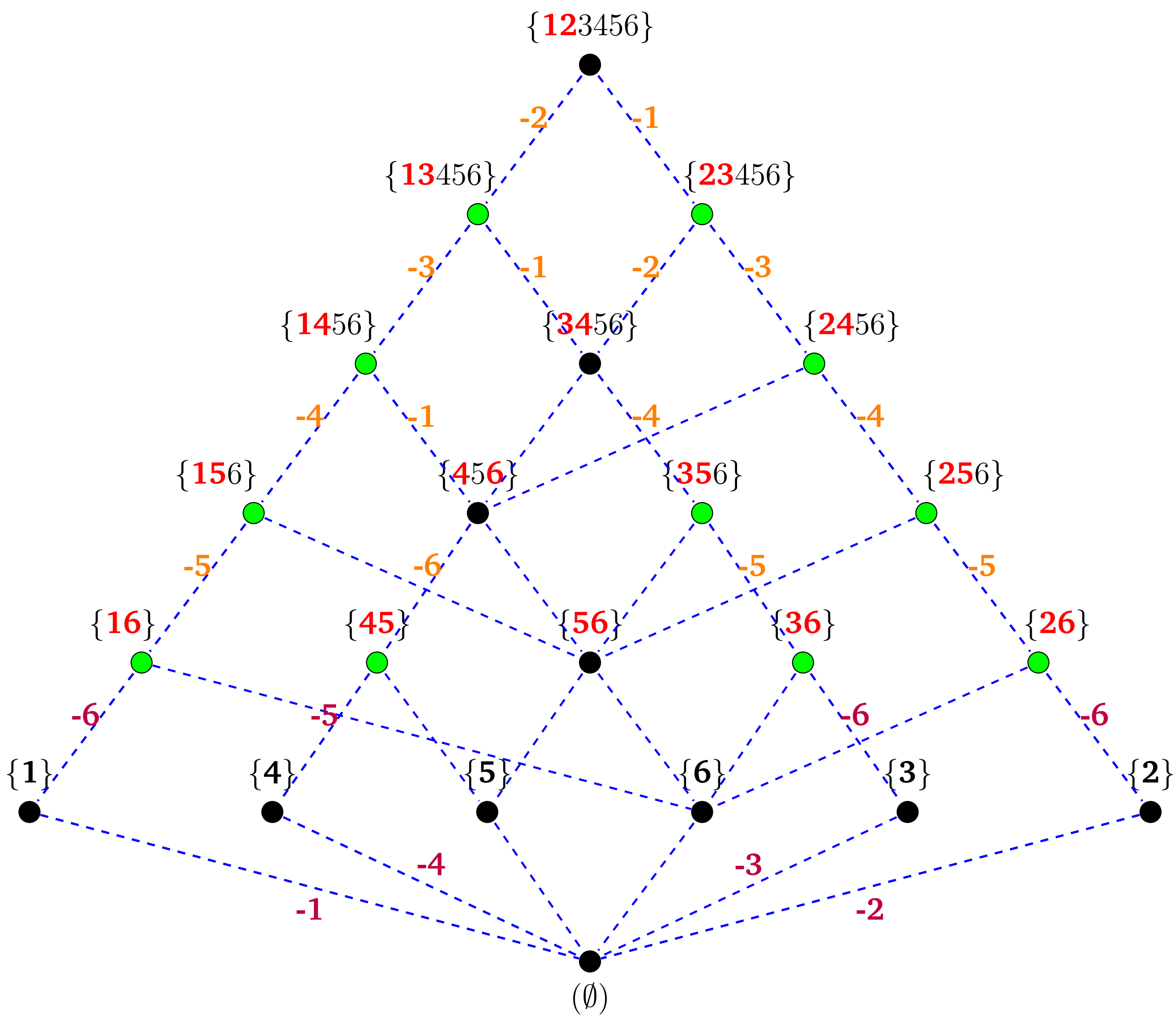

Figure 1: Lattice representation of the choice rule in Example 4

The following is a natural question.[24] For each capacity constraint q and universal set of n alternatives, does there exist a *capacity-filling* and *substitutable* choice

[24]We thank an anonymous referee for suggesting to consider this question.

rule that is not responsive, but renders a largest size MC-representation? Our next result shows that the answer is almost always in the affirmative.

Proposition 1 *For each capacity constraint $q \geq 2$ and universal set of n alternatives with $n \geq q+2$, there exists a capacity-filling and substitutable choice rule that is not responsive, but has $\binom{n-2}{q-1}$ critical sets.*

Proof. For each $q \geq 2$ and $n \geq q+2$, let $A = \{a_1, \ldots, a_n\}$. Consider the priority ordering $a_n \succ a_{n-1} \succ \cdots \succ a_1$. Let $S' = \{a_n, \ldots, a_{n-q+2}\}$. Note that $|S'| = q-1$. Since $n \geq q+2$ and $q \geq 2$, $a_1, a_2, a_3 \notin S'$. Now, consider the choice rule C such that for each $S \in \mathcal{A}$, if $S = S' \cup \{a_1, a_2\}$, then $C(S) = S' \cup \{a_1\}$; otherwise, C makes the same choice as the choice rule that is responsive to $\succ$. Note that C is *capacity-filling,* since $|S'| = q-1$ and $a_1 \notin S'$.

To see that C is *substitutable*, suppose $S_1, S_2 \in \mathcal{A}$ such that $S_1 \subsetneq S_2$. We show that $C(S_2) \cap S_1 \subset C(S_1)$. There are three cases to consider. Suppose that $S_2 = S' \cup \{a_1, a_2\}$. In this case, $C(S_2) = S' \cup \{a_1\}$. Since $S_1 \subsetneq S_2$ and $|S_2| = q+1$, $|S_1| \leq q$. Then, since C is *capacity-filling*, $C(S_1) = S_1$. Thus, $C(S_2) \cap S_1 \subset C(S_1) = S_1$. Suppose that $S_2 \neq S' \cup \{a_1, a_2\}$ and $S_1 \neq S' \cup \{a_1, a_2\}$. In this case, C is responsive at S_1 and S_2. Hence $C(S_2) \cap S_1 \subset C(S_1)$. Suppose that $S_2 \neq S' \cup \{a_1, a_2\}$ and $S_1 = S' \cup \{a_1, a_2\}$. In this case, *substitutability* can be violated only if $a_2 \in C(S_2)$. To see that this is not possible, first note that $|S_2| \geq q+2$ since $S_1 \subsetneq S_2$. Moreover, since C is responsive to $\succ$ at S_2 and a_2 is the second-worst alternative at $\succ$, we have $a_1, a_2 \notin C(S_2)$. Hence, C is *substitutable*.

Next, we show that C is not responsive. Suppose that C is responsive to a priority ordering $\succ'$. Since $a_1 \in C(S' \cup \{a_1, a_2\})$ and $a_2 \notin C(S' \cup \{a_1, a_2\})$, we have $a_1 \succ' a_2$. Since $a_1 \notin C((S' \setminus \{a_n\}) \cup \{a_1, a_2, a_3\})$ and $a_2 \in C((S' \setminus \{a_n\}) \cup \{a_1, a_2, a_3\})$, we have $a_2 \succ' a_1$, which is a contradiction. Hence, C is not responsive.

Finally, we argue that C has $\binom{n-2}{q-1}$ critical sets. To see this, first let $\mathcal{T} = \{S \in \mathcal{A} : |S| = q-1, a_1, a_2 \notin S,$ and $S \neq S'\}$. Now, for each $S \in \mathcal{T}$, since C is responsive to $\succ$ at S and a_2 is the second-worst alternative at $\succ$, we get $a_2 \in C(S \cup \{a_1, a_2\})$ and for each $a \notin S \cup \{a_1, a_2\}$, $a_2 \notin C(S \cup \{a_1, a_2, a\})$. Then, by Lemma 4, for each

$S \in \mathcal{T}$, $S \cup \{a_1\}$ is a prime atom. Note that $|\mathcal{T}| = \binom{n-2}{q-1} - 1$. Next, we show that $S' \cup \{a_2\}$ is also a prime atom of C. To see this, consider $S' \cup \{a_1, a_2\}$. First note that $S' \cup \{a_1, a_2\} \in \mathcal{M}$, since it is the largest choice set in which a_1 is chosen. Also, $a_1 \in C(S' \cup \{a_1, a_2\})$, but for each $a \notin S' \cup \{a_1, a_2\}$, since a_1 is the worst alternative at $\succ$, $a_1 \notin C(S' \cup \{a_1, a_2, a\})$. Then, by Lemma 4, $S' \cup \{a_2\}$ is also a prime atom. ■

4 Choice rules with representations of size q

Theorem 2 shows that the upper bound on the number of critical sets is achieved by responsive choice rules. On the other hand, given capacity q, for each *capacity-filling* choice rule, the minimum number of priorities that can render an MC representation is at least q. In this section, we analyze the choice rules that are q-MC. Theorem 3 shows that if the difference between the size of the universal set of alternatives and the capacity is bigger than two, then there is no *capacity-filling* choice rule that is q-MC.

To prove Theorem 3, first, we introduce a key property called *strong blocking*, and show that each *capacity-filling* and q-MC choice rule C satisfies *strong blocking*. Given a choice rule C, an alternative a **blocks** another alternative b in a choice set S if a is chosen in S and b is not chosen in S, but b is chosen when a is removed from S, i.e. if $a \in C(S)$ and $b \notin C(S)$, but $b \in C(S \setminus \{a\})$.

Strong blocking requires that if an alternative a blocks another in a choice set S that contains more than $q+1$ alternatives, then a continues to block b in any subset of S that contains more than $q+1$ alternatives including a and b. Formally, a choice rule satisfies **strong blocking** if the following is satisfied: for each choice set S with $|S| > q+1$, if an alternative a blocks another alternative b in S, then for each $S' \subset S$ with $a, b \in S'$ and $|S'| > q+1$, a blocks b in S'.

Lemma 9 *Let C be a capacity-filling choice rule. If C is q-MC, then C satisfies strong blocking.*

Proof. Consider a *capacity-filling* choice rule C that is q-MC of the priority profile

$(\succ_1, \ldots, \succ_q)$. It is easy to see that C is *substitutable*. To see that C satisfies *strong blocking*, let $S \in \mathcal{A}$ be such that $|S| > q+1$ and $a, b \in S$ be such that a blocks b in S. Since a blocks b in S, there exists $i \in \{1, \ldots, q\}$ such that $a \succ_i b$ and for each $c \in S \setminus \{a, b\}$, $b \succ_i c$. Now, let $S' \subset S$ be such that $a \in S'$ and $|S'| > q+1$. Consider the set $S'' = S' \setminus \{a\}$. Since $|S''| > q$ and b is top ranked by $\succ_i$ in S'', $b \in C(S'')$. Since C must choose q distinct alternatives from S'', there cannot be any priority that top ranks b in S'' other than $\succ_i$. Since $a \succ_i b$ and $a \in S'$, there is no priority that top ranks b in S'. It directly follows that $b \notin C(S')$, indicating that a blocks b in S'. ■

Theorem 3 *For each capacity constraint q and universal set of n alternatives, if $q > 3$ and $n > q+2$, then there is no capacity-filling choice rule that is q-MC.*

Proof. By contradiction suppose there exists a *capacity-filling* choice rule C that is q-MC, where $q > 3$ and $n > q+2$. In what follows, we show that C violates *strong blocking*. Consider the universal set of alternatives A, and let $C(A) = \{a_1, a_2, \ldots, a_q\}$. Since C is q-MC, assume w.l.o.g. that for each $i \in \{1, \cdots, q\}$, $a_i = max(A, \succ_i)$. First, we show that there are at least three distinct alternatives $\{b_1, b_2, b_3\}$ such that for each $k \in \{1, 2, 3\}$, $b_k = C(A \setminus \{a_i\}) \setminus C(A)$ for some $i \in \{1, \ldots, q\}$. To see this, by contradiction suppose there exist at most two distinct alternatives b_1 and b_2 such that for each $i \in \{1, \ldots, q\}$, $C(A \setminus \{a_i\}) \setminus C(A) \in \{b_1, b_2\}$. Since C is q-MC, it follows that for each $i \in \{1, \ldots, q\}$, b_1 or b_2 is second ranked by $\succ_i$. Since $q > 3$, by the pigeonhole principle, there exist distinct $i, j \in \{1, \ldots, q\}$ such that the same alternative, either b_1 or b_2, is second ranked by both of the associated priorities $\succ_i$ and $\succ_j$ in A. Suppose w.l.o.g. that b_1 is second ranked by both $\succ_i$ and $\succ_j$ in A. Now, consider the choice set $A \setminus \{a_i, a_j\}$. Since A is the universal set of alternatives, $|A| = n > q+2$, which implies $|A \setminus \{a_i, a_j\}| > q$. Since C is *capacity-filling*, we must have $|C(A \setminus \{a_i, a_j\})| = q$. However, we get a contradiction to this, since $C(A \setminus \{a_i, a_j\}) = \bigcup_{k \in \{1, \ldots, q\}} \max(A \backslash \{a_i, a_j\}, \succ_k)$ and $b_1 = max(A \backslash \{a_i, a_j\}, \succ_i)) = \max(A \backslash \{a_i, a_j\}, \succ_j))$, we get $|C(A \setminus \{a_i, a_j\})| = q-1$.

Now, consider the choice set $S = C(A) \cup \{b_1, b_2, b_3\}$. Since C satisfies *substitutability* and *capacity-filling*, $C(S) = C(A)$. Moreover, since $q > 3$, there exists $a_4 \in C(A) \setminus \{a_1, a_2, a_3\}$. Next, consider the choice set $S \setminus \{a_4\}$. Since C is *capacity-*

filling, $C(S \setminus \{a_4\}) \cap \{b_1, b_2, b_3\} \neq \emptyset$. Suppose w.l.o.g. that $b_1 \in C(S \setminus \{a_4\})$. It follows that a_4 blocks b_1 in S. Now, consider the choice set $S \setminus \{a_1\}$. We have $S \setminus \{a_1\} \subset S$ and $|S \setminus \{a_1\}| = q + 2 > q + 1$. But, since $b_1 \in C(A \setminus \{a_1\})$ and $S \setminus \{a_1\} \subset A \setminus \{a_1\}$, *substitutability* implies that $b_1 \in C(S \setminus \{a_1\})$. Thus, a_4 fails to block b_1 in $S \setminus \{a_1\}$, indicating that C violates *strong blocking*. By Lemma 9 this contradicts that C is q-MC. ■

If $n = q + 2$ or $n = q + 1$, then any *capacity-filling* and *substitutable* choice rule is q-MC. To see this, first suppose that $n = q + 2$. By Theorem 2, any *capacity-filling* and *substitutable* choice rule has an MC representation of a size less than or equal to q, implying that any *capacity-filling* and *substitutable* choice rule is q-MC.[25]

Now, suppose that $n = q + 1$. Let C be a *capacity-filling* and *substitutable* choice rule. Note that, by *capacity-filling*, C must reject a unique alternative from the choice set A and must accept all alternatives from any other choice set. Let $\{x\} = A \setminus C(A)$ and $A = \{a_1, \ldots, a_q, x\}$. Consider a priority profile $\pi = (\succ_1, \ldots, \succ_q)$ such that for each $i \in \{1, \ldots, q\}$, $a_i = max(A, \succ_i)$. Note that C has an MC representation of size q for the priority profile π.

Note that, given any *capacity-filling* and *substitutable* choice rule C, we are able to calculate $q^*(C)$ which is the minimum number of priority orderings required to represent C. A related question is the following: Given (n, q), what is the number $\min_{C \in H} q^*(C)$, where H is the set of all *capacity-filling* and *substitutable* choice rules? It follows from our Theorem 3 that if $q > 3$ and $n > q + 2$, then $\min_{C \in H} q^*(C) > q$. However, we leave the exact calculation of $\min_{C \in H} q^*(C)$ as an open question.

5 Conclusion

In this study, we have analyzed *capacity-filling* and *substitutable* choice rules. Despite all their eminence for applications, the size of a smallest size MC representation of a *capacity-filling* and *substitutable* choice rule and how to construct such a rep-

[25]Note that there is at least one such choice rule since any *responsive* choice rule is *capacity-filling* and *substitutable*.

resentation had been unknown. In our Theorem 1, we addressed this problem, by constructively proving that the number of critical sets determines the smallest size maximizer collecting representation for each *capacity-filling* and *substitutable* choice rule.

Our Theorem 1 together with the construction in its proof is of interest for applications in which *capacity-filling* and *substitutable* choice rules are adopted. A prominent example is the school choice problem, in which each school specifies its admission policy in the form of a *capacity-filling* and *substitutable* choice rule that reconciles the objective of admitting students with high exam scores and affirmative policies for females, ethnic minorities, or neighborhood students. A curious question is what is the simplest way of communicating the choice rule to the public. Since such a choice rule can be represented as an MC choice rule, it is natural to assume that as the size of this representation decreases, the communication can be easier. Our Theorem 1 and the related construction in its proof shows us how to choose the priority orderings so as to obtain a smallest size MC representation.

Well-known examples of *capacity-filling* and *substitutable* choice rules are responsive choice rules. In Theorem 2, we show that the size of the smallest size MC representation rendered by responsive choice rules is largest among all *capacity-filling* and *substitutable* choice rules. We have also analyzed q-MC choice rules that are easy to communicate. An impossibility result follows from this characterization: if the difference between the size of the universal set of alternatives and the capacity is bigger than two, then there is no q-MC choice rule. This result and Theorem 2 indicate that using an MC representation may not be the easiest way to communicate every *capacity-filling* and *substitutable* choice rule.

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