Lexicographic choice under variable capacity constraints

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joint with...

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school matching problem

- a set of students with preferences over the schools (R)
- a set of schools with capacity constraints (q)
- a matching problem is a pair (R, q), consisting of preferences and capacities.

a mechanism associates each matching problem with an allocation respecting capacity constraints.

*** choice rules of the schools are left as a part of the design.

link btw. choice and matching

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Question: What are the other implications of lexi. choice for the resulting matching under DA algorithm?

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- We provide an axiomatic characterization of lexicographic choice rules under variable capacity constraints.
- We provide a characterization of deferred acceptance mechanisms that operate based on a lexicographic choice structure, instead of a priority structure.
- We present some implications related to a debate about Boston school choice system.

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classical axioms

Acceptance: An alternative is rejected from a choice set only if the capacity is full.

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Substitutes: If an alternative is chosen from a choice set at a capacity, then it is also chosen from any subset of the choice set that contains the alternative at the same capacity.

$$\forall (S,q), \text{if } a \in C(S,q) \text{ and } b \in S, \text{ then } a \in C(S\backslash\{b\}, q).$$

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Monotonicity: If an alternative is chosen from a choice set at a capacity, then it is also chosen from the same choice set at any higher capacity.

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a new property: CWARP

- An alternative a is revealed to be preferred to an alternative b at a capacity $q > 1$ if there is a choice set S s.t.

  1. a and b are both rejected when capacity is $q-1$, and
  2. when capacity is q, a is chosen but b is not chosen.

That is: $a \; R_q \; b$ if there exists $S \in \mathcal{A}$ s.t.

1. $a, b \notin C(S, q-1)$, and
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Capacity-wise weak axiom of revealed preference (CWARP): For each capacity $q > 1$ and each pair $a, b \in A$, if a is revealed preferred to b at q, then b is not revealed preferred to a at q.

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CWARP vs. WARP

- CWARP is a counterpart of (WARP)
- In contrast to WARP, CWARP requires consistency of the choice behavior in responding to changes in the choice set together with changes in the capacity.

Proposition: *A choice rule satisfies acceptance, substitutes, monotonicity, and CWARP if and only if it is lexicographic.*

feasibility constraints

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- Matching with contracts: alternatives are (agent, contract) pairs, where each agent can be chosen only with a single contract. (Kelso and Crawford (1982), Hatfield and Milgrom, (2005))

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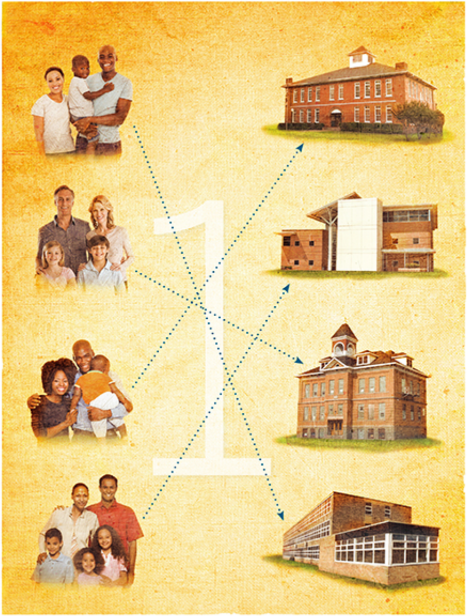
Proposition *A feasibility-constrained choice rule is lexicographic if and only if it satisfies acceptance*, monotonicity, and the CSARP.*



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Proposition: *A mechanism is a lexicographic deferred acceptance mechanism iff it satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, strategy-proofness, and demand-monotonicity.*

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The Boston School Choice System

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At each school,

- There are some neighborhood students.
- There are two different priority orderings: a walk-zone priority ordering, which gives priority to the neighborhood students over all the other students, and an open priority ordering which does not give priority to any student for being a neighborhood student.
- The Boston school district aims to assign half of the school seats based on the walk-zone priority ordering and the other half based on the open priority ordering.

How to Achieve Diversity in Boston

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Dur et al. compare the four choice rules in terms of how much they are biased for or against the neighborhood students.

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Observation:

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- *Rotating choice rule is the only choice rule among the four that satisfies CWARP. Hence the only one that has lexicographic representation under variable capacity constraints.*

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responsive choice

Let $\succ$: $a_1 \succ a_2 \cdots \succ a_n$ be a priority ordering over alternatives.

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For ex., if $q = 3$, then

$$C(a_1, a_5, a_6, a_7) = \{a_1, a_5, a_6\} \text{ \& } C(a_1, a_n) = \{a_1, a_n\}$$

.