Implementation via Codes of Rights

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Pareto Rule with Minimal Liberalism (PwL):

- a society, $N = \{1, 2, 3\}$, with individual preferences on $a$, $b$, $c$
- for each alternative $x \neq a$, $x$ is (socially) acceptable iff $x$ is Pareto efficient
- $a$ is acceptable iff $a$ is not only Pareto efficient, but also agent 1 prefers $a$ to $b$
implementation problem

How can we shape an interaction in the society to rule out unacceptable outcomes and implement only the acceptable ones?
How can we shape an *interaction* in the society to rule out unacceptable outcomes and implement *only* the acceptable ones?
our approach

to implement PwL, for each \( x, y \in \{a, b, c\} \)

- entitle \( N \) to move from \( x \) to \( y \)

- additionally, entitle 1 to move from \( a \) to \( b \)
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- real life mechanisms: constitutions, legal codes, social norms, corporate culture
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- implementation theory: we design game forms by adhering to well known equilibrium concepts
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- real life mechanisms: constitutions, legal codes, social norms, corporate culture

- implementation theory: we design game forms by adhering to well known equilibrium concepts

observation: the resulting general game forms are quite different from (more complicated than) the real life mechanisms that we observe
we aim to formulate and analyze a framework for implementation written in a language closer to the institutional real life mechanisms
Nash implementation is the benchmark model for classical implementation

- used mechanisms are complicated
- characterization conditions are not easy to verify
- asymmetric conditions apply for two agents
- many interesting rules are not Nash-implementable
Nash implementation is the benchmark model for classical implementation

► used mechanisms are complicated

► characterization conditions are not easy to verify

► asymmetric conditions apply for two agents

► many interesting rules are not Nash-implementable
Subgame Perfect Nash implementation, Moore & Repullo (1988)

Undominated Nash implementation, Palfrey & Srivastava (1991)

Virtual implementation, Abreu & Sen (1991)

Implementation with Partially Honest Individuals, Dutta & Sen (2012)
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A persistent criticism: used mechanisms are difficult to interpret
\( A \) is a non empty finite set of alternatives

\( N \) is the set of agents with \( n \) elements

each non empty \( K \subset N \) is a coalition

\( u_i \) denotes the preference relation of agent \( i \)

\( u \) is a preference profile of \( N \) on \( A \)
A is a non empty finite set of alternatives

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each non empty \( K \subset N \) is a coalition

\( u_i \) denotes the preference relation of agent \( i \)

\( u \) is a preference profile of \( N \) on \( A \)

a social choice rule, \( F \), maps each \( u \) to a set of acceptable alternatives \( F(u) \)
Nash implementation

design object is a game form \((M, g)\)
Nash implementation

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\(M_i\) is the strategy (message) set of agent \(i\), and \(M = \Pi_{i \in N} M_i\)

an outcome function \(g : M \rightarrow A\)
Nash implementation

design object is a game form $(M, g)$

$M_i$ is the strategy (message) set of agent $i$, and $M = \prod_{i \in N} M_i$

an outcome function $g : M \rightarrow A$

an SCR $F$ is Nash implementable via a game form $(M, g)$ if

for each $u \in \mathcal{P}$, $F(u) = \{g(m) : m \in NE(M, g, u)\}$
a rights structure $\Gamma = (S, h, \gamma)$
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- $S$ is the set of (social) states
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- an outcome function $h : S \rightarrow A$
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- $S$ is the set of (social) states
- an outcome function $h : S \rightarrow A$
- a code of rights $\gamma$ maps each (distinct) state pair $(s, t)$ to a coalition family

a coalition $K \in s \xrightarrow{\gamma} t$ is interpreted as $K$ is given the right to move from $s$ to $t$
rights structure, \( \Gamma = (S, h, \gamma) \)

\( S \) is the set of (social) states:
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i) a state can be an alternative ($S = A$), then

a coalition $K \in a \xrightarrow{\gamma} b$ is interpreted as $K$ can change the status quo from $a$ to $b$

ii) a state can be a proposal for an alternative with some supporting evidence, then
rights structure, $\Gamma = (S, h, \gamma)$

$S$ is the set of (social) states:

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ii) a state can be a proposal for an alternative with some supporting evidence, then

\[ a \text{ coalition } K \in s \xrightarrow{\gamma} t \text{ is interpreted as } K \text{ can refute the evidence for } s \]
Given a pair \((S, h)\) and a preference profile \(u\), for each distinct \((s, t)\), a coalition \(K\) prefers to move from \(s\) to \(t\), denoted by \(K \in s \xrightarrow{u} t\), iff

\[
\text{for each } i \in K, h(t) u_i h(s)
\]
Definition: For each $u \in \mathcal{P}$, a state $s$ is an equilibrium of rights structure $\Gamma$ at $u$ if for each $t \in S$,

$$(s \gamma \rightarrow t) \cap (s \overset{u}{\rightarrow} t) = \emptyset$$
Definition: For each $u \in \mathcal{P}$, a state $s$ is an equilibrium of rights structure $\Gamma$ at $u$ if for each $t \in \mathcal{S}$,

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let $E(\Gamma, u)$ denote this equilibrium set
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**Definition:** For each $u \in \mathcal{P}$, a state $s$ is an equilibrium of rights structure $\Gamma$ at $u$ if for each $t \in S$,

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let $E(\Gamma, u)$ denote this equilibrium set

**Definition:** $F$ is $\Gamma$-implementable if there is a rights structure $\Gamma = (S, h, \gamma)$ such that,

for each $u \in \mathcal{P}$, $F(u) = h(E(\Gamma, u))$
let $N = \{1, 2, 3\}$ and $A = \{I, G\}$

$F(u) = I$ iff $I$ is preferred to $G$ by at least two agents

let $S = \{I^{1,2}, I^{1,3}, I^{2,3}, G^{1,2}, G^{1,3}, G^{2,3}\}$
Let $N = \{1, 2, 3\}$ and $A = \{I, G\}$.

$F(u) = I$ iff $I$ is preferred to $G$ by at least two agents.

Let $S = \{I^{1,2}, I^{1,3}, I^{2,3}, G^{1,2}, G^{1,3}, G^{2,3}\}$.

Let $\gamma$ be such that, for each $i, j \in N$, and $s \in S$,

$I^{i,j} \xrightarrow{\gamma} s = \{\{i\}, \{j\}\}$

$G^{i,j} \xrightarrow{\gamma} s = \{\{i\}, \{j\}\}$
suppose the true preference profile is $u^*$, so $F(u^*) = I$
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$$
\begin{array}{ccc}
I^{2,3} & \rightarrow & G^{2,3} \\
1 & \rightarrow & G^{1,3} \\
& \rightarrow & G^{1,2} \\
\end{array}
$$

$$
\begin{array}{ccc}
1 & 2 & 3 \\
G & I & I \\
I & G & G \\
\end{array}
$$
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$\begin{array}{c}
I^{2,3} \\
G^{2,3} & G^{1,3} & G^{1,2}
\end{array}$

$\begin{array}{c|c|c}
& 1 & 2 & 3 \\
\hline
G & I & I \\
I & G & G
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\[
\begin{array}{c}
I_{2,3} \\
3 & 2 \end{array}
\]

\[
\begin{array}{c}
G_{2,3} \\
G_{1,3} \\
G_{1,2} \\
\end{array}
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\[
\begin{array}{ccc}
  & 1 & 2 & 3 \\
 1 & G & I & I \\
 2 & I & G & G \\
\end{array}
\]
Nash Implementation:

- **a mechanism**: $(M, g)$
  
  \[
g : M \rightarrow A
  \]

  \[
  F(u) = NE(M, g, u)
  \]

Γ-implementation:

- **a rights structure** $\Gamma = (S, h, \gamma)$

  \[
h : S \rightarrow A
  \]

  \[
  F(u) = E(\Gamma, u)
  \]
Nash Implementation:

a game form: \((M, g)\)

\[ g : M \rightarrow A \]

\[ F(u) = NE(M, g, u) \]

\[ NE(M, g, u) \subseteq \Gamma \]

\[ F(u) = E(\Gamma, u) \]

\[ F(u) \triangleq E(\Gamma, u) \]

\[ \Gamma \text{-implementation:} \]

a rights structure \(\Gamma = (S, h, \gamma)\)

\[ h : S \rightarrow A \]

\[ E(\Gamma, u) \triangleq F(u) \]
Nash Implementation:

a game form: \((M, g)\)

\(g : M \to A\)

\(F(u) = \text{NE}(M, g, u)\)

\[\text{let } S = M \text{ and } h = g\]

\(s \xrightarrow{\gamma} t = \begin{cases} \{i\} & \text{if } t = (t_i, s_{-i}) \\ \emptyset & \text{otherwise} \end{cases}\)

\[\Gamma\text{-implementation:}\]

a rights structure \(\Gamma = (S, h, \gamma)\)

\(h : S \to A\)

\(F(u) = E(\Gamma, u)\)
Nash Implementation:

a game form: \((M, g)\)

\(g : M \rightarrow A\)

\(F(u) = NE(M, g, u)\)

\(\Gamma\)-implementation:

a rights structure \(\Gamma = (S, h, \gamma)\)

\(h : S \rightarrow A\)

\(F(u) = E(\Gamma, u)\)

let \(S = M\) and \(h = g\)

\[ s \xrightarrow{\gamma} t = \begin{cases} \{i\} & \text{if } t = (t_i, s_{-i}) \\ \emptyset & \text{otherwise} \end{cases} \]
PwL is $\Gamma$-implementable, let $S = A$ and $h$ be the identity map, design $\gamma$ s.t. for each $x \neq a$ and $y \in A$,

- $x \xrightarrow{\gamma} y = \{N\}$
- $a \xrightarrow{\gamma} b = \{\{1\}, N\}$
PwL is not Nash implementable

let $u$ and $u'$ be as follows, where $F(u) = \{b, c\}$ and $F(u') = \{b\}$.

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1. Γ-implementation
2. Γ-implementation with external stability
3. Γ underlying Nash implementability
4. γ-implementation
5. Γ-implementation with a minimal state space
Γ-implementation
Suppose that alternative \( a \) is acceptable at \( u^1 \) according to the SCR in question. Then, if \( a \) does not fall in anyone’s ranking relative to any other alternative in going from profile \( u^1 \) to profile \( u^2 \), \textit{monotonicity} requires that \( a \) also be acceptable at \( u^2 \).
Suppose that alternative $a$ is acceptable at $u^1$ according to the SCR in question. Then, if $a$ does not fall in anyone’s ranking relative to any other alternative in going from profile $u^1$ to profile $u^2$, monotonicity requires that $a$ also be acceptable at $u^2$.

However, if $a$ does fall relative to some $b$ in someone’s ranking, monotonicity imposes no restriction.
$F$ is monotonic if for each $u^1, u^2 \in \mathcal{P}$, and $a \in F(u^1)$, we have $a \in F(u^2)$ whenever for every $i \in N$,

$$\{ b \in A : a u^1_i b \} \subseteq \{ c \in A : a u^2_i c \}$$
monotonicity and implementation

for an SCR $F$ let $I(F) = \{ a \in A : a \in F(u) \text{ for some } u \}$

$F$ is image monotonic if for each $u^1, u^2 \in \mathcal{P}$, and $a \in F(u^1)$, we have $a \in F(u^2)$ whenever for every $i \in N$,

$$\{ b \in I(F) : a \ u_i^1 \ b \} \subseteq \{ c \in A : a \ u_i^2 \ c \}$$
monotonicity and implementation

$F$ is image monotonic if for each $u^1, u^2 \in \mathcal{P}$, and $a \in F(u^1)$, we have $a \in F(u^2)$ whenever for every $i \in N$,

$$\{ b \in I(F) : a \ u_i^1 \ b \} \subseteq \{ c \in A : a \ u_i^2 \ c \}$$

**Proposition:** Given an SCR $F$, the following are equivalent,

i. $F$ is image monotonic

ii. $F$ is $\Gamma$ – implementable
monotonicity and implementation

$F$ is **image monotonic** if for each $u^1, u^2 \in \mathcal{P}$, and $a \in F(u^1)$, we have $a \in F(u^2)$ whenever for every $i \in N$,

$$\{b \in I(F) : a u^1_i b\} \subseteq \{c \in A : a u^2_i c\}$$

**Proposition:** Given an SCR $F$, the following are equivalent,

i. $F$ is image monotonic

ii. $F$ is $\Gamma$ – implementable
(IB): $\Gamma$ is individual based if for each distinct $s, t \in S$, $s \overset{\gamma}{\rightarrow} t$ is either empty or consists of only single agents.

**Proposition:** Given an SCR $F$, the following are equivalent,

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**Proposition:** Given an SCR $F$, the following are equivalent,

i. $F$ is image monotonic

ii. $F$ is $\Gamma$ – implementable

iii. $F$ is $\Gamma$ – implementable via an IB rights structure.
a direct rights structure $\Gamma^d$ is a triple $(S^d, h^d, \gamma^d)$ where;

$$S^d = \{(a, u) : a \in F(u)\}$$
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$$S^d = \{(a, u) : a \in F(u)\}$$

- for each $(a, u) \in S^d$, $h^d(a, u) = a$
a direct rights structure $\Gamma^d$ is a triple $(S^d, h^d, \gamma^d)$ where;

$$S^d = \{ (a, u) : a \in F(u) \}$$

- for each $(a, u) \in S^d$, $h^d(a, u) = a$
- for each $(a, u), (b, v) \in S^d$, and $i \in N$,

$$\{ i \} \in (a, u) \xrightarrow{\gamma^d} (b, v) \text{ iff } a \, u_i \, b$$
example: majority rule

- let $N = \{1, 2, 3\}$ and $A = \{a, b\}$
- there are four profiles $\{u^0, u^3, u^2, u^1\}$ where $a$ is chosen since $a$ is preferred to $b$ by $N$, $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ respectively
- similarly let $\{v^0, v^3, v^2, v^1\}$ be the profiles where the roles of $a$ and $b$ are changed, so $b$ is chosen

consider the direct rights structure $\Gamma^d$ tailored for this rule, where

$$S^d = \{(a, u^m), (b, v^m)\}_{m=1}^4$$
suppose the true preference profile is $u^1$, so $F(u^1) = a$
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\begin{center}
\begin{tabular}{ccc}
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$b$ & $a$ & $a$
\end{tabular}
\end{center}
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suppose the true preference profile is \( u^1 \), so \( F(u^1) = a \)
External stability of rights structures
Consider the rights structure $\Gamma$ at $u$, where

- **blue**: equilibrium states
- **red**: neq-states with unacceptable outcomes
- **green**: neq-states with acceptable outcomes
In general, there may not be a path connecting a non-equilibrium state to an equilibrium state.

- **blue**: equilibrium states
- **red**: neq-states with unacceptable outcomes
- **green**: neq-states with acceptable outcomes
If the $\Gamma$-implementable SCR $F$ is also unanimous, then always there exists such a path

- **blue**: equilibrium states
- **red**: neq-states with unacceptable outcomes
- **green**: neq-states with acceptable outcomes
A rights structure, $\Gamma = (S, h, \gamma)$ is **externally stable** if for each $u \in \mathcal{P}$, and for each $s \in S$ with an unacceptable outcome, there exists $t \in E(\Gamma, u)$ such that, $(s \xrightarrow{\gamma} t) \cap (s \xrightarrow{u} t) \neq \emptyset$.

- **Blue:** equilibrium states
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a rights structure, $\Gamma = (S, h, \gamma)$ is externally stable if for each $u \in P$, and for each $s \in S$ with an unacceptable outcome, there exists $t \in E(\Gamma, u)$ such that, $(s \xrightarrow{\gamma} t) \cap (s \xrightarrow{u} t) \neq \emptyset$

**Diagram:**

- **blue:** equilibrium states
- **red:** neq-states with unacceptable outcomes
- **green:** neq-states with acceptable outcomes
$F$ is **winner monotonic** if for each $u^1, u^2 \in \mathcal{P}$, and $a \in F(u^1)$, we have $a \in F(u^2)$ whenever for every $i \in N$,

$$\{ b \in F(u^2) : a \ u^1_i \ b \} \subseteq \{ c \in A : a \ u^2_i \ c \}$$
externally stable rights structures

$F$ is **winner monotonic** if for each $u^1, u^2 \in \mathcal{P}$, and $a \in F(u^1)$, we have $a \in F(u^2)$ whenever for every $i \in N$,

$$\{b \in F(u^2) : a u^1_i b\} \subseteq \{c \in A : a u^2_i c\}$$

**Proposition:** $F$ is implementable via an externally stable rights structure if and only if $F$ is **winner monotonic**.
Rights structures underlying the Nash implementability
for a rights structure $\Gamma = (S, h, \gamma)$,

(\textbf{IB}) $\Gamma$ is \textit{individual based} if for each distinct $s, t \in S$, $s \xrightarrow{\gamma} t$ is either empty or consists of only single agents.
Γ underlying the Nash implementability

for a rights structure \( \Gamma = (S, h, \gamma) \),

\textbf{(IB)} \( \Gamma \) is individual based if for each distinct \( s, t \in S \), \( s \xrightarrow{\gamma} t \) is either empty or consists of only single agents.

\textbf{(IT)} \( \Gamma \) is individually transitive if for each \( s, t, w \in S \), and \( i \in N \),

\[ \text{if } \{i\} \in (s \xrightarrow{\gamma} t) \cap (t \xrightarrow{\gamma} w), \text{ then} \]

\[ s \xrightarrow{\{i\}} t \xrightarrow{\{i\}} w \]
for a rights structure $\Gamma = (S, h, \gamma)$,

**(IB)** $\Gamma$ is **individual based** if for each distinct $s, t \in S$, $s \xrightarrow{\gamma} t$ is either empty or consists of only single agents.

***(IT)*** $\Gamma$ is **individually transitive** if for each $s, t, w \in S$, and $i \in N$,

\[
\text{if } \{i\} \in (s \xrightarrow{\gamma} t) \cap (t \xrightarrow{\gamma} w), \text{ then } \{i\} \in s \xrightarrow{\gamma} w
\]
Γ underlying the Nash implementability

**Proposition:** Given $N \geq 3$, $F$ is Nash-implementable iff $F$ is implementable via an individual based and individually transitive rights structure, $\Gamma$. 
(⇐) given \((S, h, \gamma)\), each \(i \in N\), announces a state, an agent and a positive integer

let outcome function \(g\) be s.t. for each joint strategy \(m\),

(1) if everybody announces the same state \(s\), then we have \(g(m) = h(s)\).
(⇐) given \( (S, h, \gamma) \), each \( i \in N \), announces a state, an agent and a positive integer

let outcome function \( g \) be s.t. for each joint strategy \( m \),

(1) if everybody announces the same state \( s \), then we have \( g(m) = h(s) \).

(2) if everybody other than \( i \) announces the state \( s \) and \( i \) announces a state \( s' \). If \( \{i\} \in s \xrightarrow{\gamma} s' \), then \( g(m) = h(s') \). If \( \{i\} \not\in s \xrightarrow{\gamma} s' \), then \( g(m) = h(s) \).
(⇐) given \((S, h, \gamma)\), each \(i \in N\), announces a state, an agent and a positive integer

let outcome function \(g\) be s.t. for each joint strategy \(m\),

(1) if everybody announces the same state \(s\), then we have \(g(m) = h(s)\).

(2) if everybody other than \(i\) announces the state \(s\) and \(i\) announces a state \(s'\). If \({i}\) \(\in s \xrightarrow{\gamma} s'\), then \(g(m) = h(s')\). If \({i}\) \(\not\in s \xrightarrow{\gamma} s'\), then \(g(m) = h(s)\).

(3) if at least three different states are announced, then let \(i\) be the agent who announces the highest integer.

Let agent \(j\) be the agent whom is indicated by agent \(i\).

Let \(s''\) is the state announced by agent \(j\). We have, \(g(m) = h(s'')\).
design object is a \textit{dc-mechanism} \((M, D, g)\)
A design object is a **dc-mechanism** \((M, D, g)\)

- \(M_i\) is the strategy set of agent \(i\) and \(M = \Pi_{i \in N} M_i\)

- an outcome function \(g : M \rightarrow A\)
A design object is a \textit{dc-mechanism} \((M, D, g)\)

\(M_i\) is the strategy set of agent \(i\) and \(M = \prod_{i \in N} M_i\)

an outcome function \(g : M \rightarrow A\)

for each agent \(i\), \(D_i\) maps each joint strategy \(m\) to a subset of \(M_i\), i.e. \(D_i : M \rightarrow M_i\).

In a dc-mechanism, if an agent \(i\) would deviate from strategy \(m\), he is constrained to choose his strategy from \(D_i(m)\).
a joint strategy \( m \) is an equilibrium of \((M, \mathcal{D}, g)\) at \( u \) iff for each \( i \in N \) and \( m'_i \in \mathcal{D}_i(m) \),

\[
g(m) \ u_i \ g(m'_i, m_{-i})
\]

we denote the equilibria of \((M, \mathcal{D}, g)\) at \( u \), by \( \mathbf{E}(M, \mathcal{D}, g, u) \).

**Proposition:** Given \( N \geq 3 \), \( F \) is \( \Gamma \)-implementable if and only if there exists a dc-mechanism, \((M, \mathcal{D}, g)\), such that for each \( u \), \( F(u) = \mathbf{E}(M, \mathcal{D}, g, u) \).
A SOCIAL EQUILIBRIUM EXISTENCE THEOREM*

BY GERARD DEBREU

COWLES COMMISSION FOR RESEARCH IN ECONOMICS

Communicated by J. von Neumann, August 1, 1962

In a wide class of social systems each agent has a range of actions among which he selects one. His choice is not, however, entirely free and the actions of all the other agents determine the subset to which his selection is restricted. Once the action of every agent is given, the outcome of the social activity is known. The preferences of each agent yield his complete ordering of the outcomes and each one of them tries by choosing his action in his restricting subset to bring about the best outcome according to his
\( \gamma \)-implementation
we specify $S = A$ and $h$ is the identity map

**Definition:** For each $u$, an alternative $a$ is a $\gamma$-equilibrium at $u$ if

for each $b \in A$, $(a \xrightarrow{\gamma} b) \cap (a \xrightarrow{u} b) = \emptyset$
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**Definition:** $F$ is $\gamma$-implementable if there is a code of rights, $\gamma$, such that

$$\text{for each } u \in \mathcal{P}, \ F(u) = E(\gamma, u)$$
Consider any $2 \times 2$ pure exchange economy where agents have monotonic, continuous and strictly convex preferences over the entire consumption space $\mathbb{R}^2_+$.

Given an endowment vector $\omega \gg 0$, let $F^\omega$ be the rule that chooses the Walrasian allocations at each preference profile.

for given $\omega$, for each $a, b \in \mathbb{R}^2_+$, let

$$a \overset{\gamma}{\rightarrow} b = \begin{cases} \{1\}, \{2\} & \text{if } a, b \text{ and } \omega \text{ are collinear or } a \text{ is not feasible}, \\ \emptyset & \text{otherwise} \end{cases}$$
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\[ a \xrightarrow{\gamma} b = \begin{cases} \{1\}, \{2\} & \text{if } a, b \text{ and } \omega \text{ are collinear or } a \text{ is not feasible}, \\ \emptyset & \text{otherwise} \end{cases} \]
Proposition: $F$ is $\gamma$-implementable if and only if $F$ satisfies Maskin monotonicity and binary consistency.
Consider any \( u \in P \) and \( a, b \in A \) such that \( b \) is not Pareto dominated by \( a \) at \( u \). Let \( u^{ab} \) stand for the profile obtained from \( u \) such that the only difference is for each \( c \in A \setminus \{a, b\} \) and \( i \in N \), we have \( a u_i^{ab} c \) and \( b u_i^{ab} c \).
A function $F$ satisfies **binary consistency** if for each $u \in \mathcal{P}$ and $a \in A$, we have $a \in F(u)$ whenever for each $b \in A$ that is not Pareto dominated by $a$ at $u$, one has $a \in F(u^{ab})$.

**Proposition:** $F$ is $\gamma$-implementable if and only if $F$ satisfies **Maskin monotonicity** and **binary consistency**.
State complexity of rights structures
A critical rights structure $\Gamma^c$ is a triple $(S^c, h^c, \gamma^c)$ where:

\[
S^c = \{ (a, u) : u \in \Lambda(F, a) \}
\]

- For each $(a, u) \in S^c$, $h^c(a, u) = a$
- For each $(a, u), (b, v) \in S^c$, a coalition $K \in (a, u) \xrightarrow{\gamma^c} (b, v)$ iff there exists $i \in K$ s.t. $a \mathrel{u_i} b$
for each \( a \in A \), a preference profile \( u \) is critical for \( a \), if for any \( b \), any agent \( i \) with \( a \prec_i b \) reverses his preference between \( a \) and \( b \) for any \( b \), then \( F \) no longer chooses \( a \) in the new profile.
for each $a \in A$, a preference profile $u$ is critical for $a$, if for any $b$, any agent $i$ with $a \ u_i \ b$ reverses his preference between $a$ and $b$ for any $b$, then $F$ no longer chooses $a$ in the new profile.

Ex: Consider the majority rule and let $u$ and $u'$ be as follows,

\[
\begin{array}{ccc}
1 & 2 & 3 \\
a & a & c \\
c & c & b \\
b & b & a \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
1 & 2 & 3 \\
a & a & c \\
c & b & a \\
b & c & b \\
\end{array}
\]

$u$ is critical whereas $u'$ is not.
Proposition: Given \((S, h)\), if an IB rights structure \((S, h, \gamma)\) implements \(F\), then the number of states in \(S\) is at least equal to the number of states in \(S^c\), i.e. \(|S| \geq |S^c|\).
related literature

rights structures
- Sen (1960)
- Sertel (2001)
- Ben and Sugden (2011)

da-mechanisms
- Debreu (1952)
- Glazer and Rubinstein (2012)

effectivity fncs. and imp.
- Peleg and et. al.

external stability
- Greenberg (1990)
- Chwe (1994)
Our analysis of Γ-implementation shows that,

- we can implement SCRs by explicitly designing simple rights structures
- why complicated game form are inevitable in classical implementation.