On Capacity-filling and Substitutable Choice Rules

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joint with...

Motivation

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- A well-known example is the school choice problem in which each school has a certain number of seats to be allocated among students.
- Endowing each school with a choice rule is a part of the design process.

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Capacity-filling & Substitutes

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Substitutes: If an alternative is chosen from a choice set, then it is also chosen from any subset of the choice set that contains the alternative.

$$\text{If } a \in C(S) \text{ and } b \in S, \text{ then } a \in C(S \setminus \{b\}).$$

Substitutes is crucial because...

- Substitutable choice rules have been a standard tool following the seminal work of Kelso and Crawford, 1982, in broadening the classical matching model with single priority.
- Hatfield and Milgrom, 2005 show that substitutability guarantees the existence of stable matchings.
- Hatfield and Kojima, 2006 show that substitutability is almost necessary for the non-emptiness of the core in allocations problems
- Similarly, several classical results of matching literature have been generalized with substitutable choice rules (Roth and Sotomayor, 1990; Hatfield and Milgrom, 2005).

Ex 1: responsive choice

Let $\succ$: $a_1 \succ a_2 \cdots \succ a_n$ be a priority ordering over alternatives.

a choice rule C is responsive to $\succ$ if for each $S \in \mathcal{A}$, $C(S)$ is obtained by choosing the highest $\succ$-priority alternatives until the capacity q is reached or no alternative is left.

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For ex., if $q = 3$, then

$$C(a_1, a_5, a_6, a_7) = \{a_1, a_5, a_6\} \ \& \ C(a_1, a_n) = \{a_1, a_n\}$$

.

Ex 2: lexi(cographic) choice

a priority profile $\pi = (\succ_1, \ldots, \succ_q)$ is an ordered list of q priorities.

a choice rule C is lexicographic for π if for each $S \in \mathcal{A}$, $C(S)$ is obtained by

- choosing the highest $\succ_1$-priority alternative in S,
- then choosing the highest $\succ_2$-priority alternative among the remaining ones, and so on until q.

What do we know from the literature?

- capacity-filling and substitutes implies path independence *(Plott, 1973)*, which requires

$$\forall S, S' \in \mathcal{A}, C(S \cup S') = C(C(S) \cup C(S'))$$

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$$\forall S, S' \in \mathcal{A}, C(S \cup S') = C(C(S) \cup C(S'))$$

- It follows from Aizerman and Malishevski, 1981 that each capacity-filling and subsitutable choice rule has a collected maximal representation.

collected maximal choice

a choice rule C has a collected maximal representation of size $m \in \mathbb{N}$ if there is a list of m priority orderings $(\succ_1, \ldots, \succ_m)$ such that

i. $\forall S \in \mathcal{A}$ with $|S| \leq q$, $C(S) = S$, and

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ii. $\forall S \in \mathcal{A}$ with $|S| > q$, $C(S)$ is obtained by collecting the maximizers of the priority orderings in S, i.e.

$$C(S) = \bigcup_{i \in \{1,\ldots,m\}} \max(S, \succ_i).$$

Example of C-max rule

Let $A = \{1, 2, 3, 4, 5, 6\}$ and consider the priority profile $(\succ_\alpha, \succ_\beta, \succ_\gamma, \succ_\delta)$. Let C be the 2-capacity-filling choice rule that is collected maximal of this priority profile.

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
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6	6	6	6
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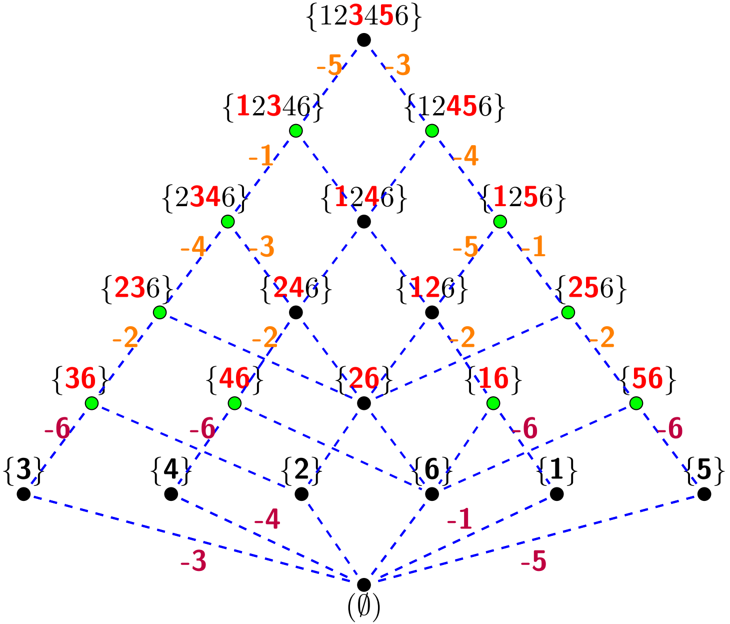
spoiler!!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Definition: A choice set $S \in \mathcal{A}$ is critical for C if $|S| = q$, and there exists $a \notin S$ such that

- a is chosen from $S \cup \{a\}$,
- and a is no longer chosen whenever any other element is added to $S \cup \{a\}$.

Ex: critical sets of responsive choice

Suppose C is responsive to $\succ$: $a_1 \succ a_2 \cdots \succ a_n$

Q: What are the critical sets?

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Q: What are the critical sets?

A: Each choice set with q alternatives, that includes a_n and excludes a_{n-1}.

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We show that the number of critical sets determines the smallest size collected maximal representation.

Theorem 1: *If a choice rule C satisfies capacity-filling and substitutes, then*

i. *C has a collected maximal representation of the size equal to the number of its critical sets.*

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Theorem 1: *If a choice rule C satisfies capacity-filling and substitutes, then*

i. *C has a collected maximal representation of the size equal to the number of its critical sets.*

ii. *C fails to have a collected maximal representation of any size smaller than the number of its critical sets.*

Proof

Proposition *The upper bound on the number of critical sets is* $\binom{n-2}{q-1}$ *and it is achieved by responsive choice rules.*

Put differently, responsive choice rules render a collected maximal representation of the largest size among all q-capacity-filling choice rules that satisfy substitutes.

collected q-maximal choice

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- schools typically come up with *slot specific priorities* and apply these sequentially
- If a school with q-many slots chooses according to a collected q-maximal choice rule, then
  - **i.** school can transparently reveal the order to be maximized,
  - **ii.** choice is immune to, rather debatable, order based affirmative action effects.

an impossibility

Theorem 2: *For each capacity constraint q and universal set of alternatives with n members, if $q > 3$ and $n > q + 2$, then there is no capacity-filling choice rule C that is collected q-maximal.*

conclusion

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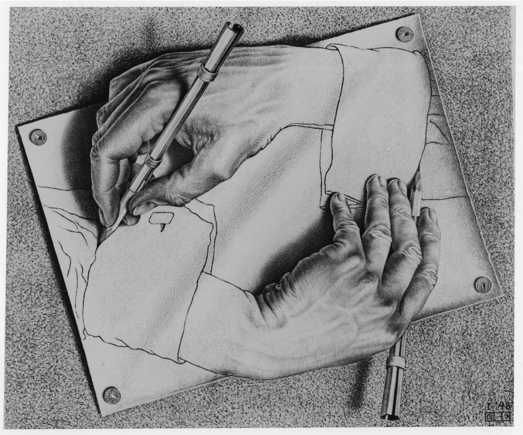

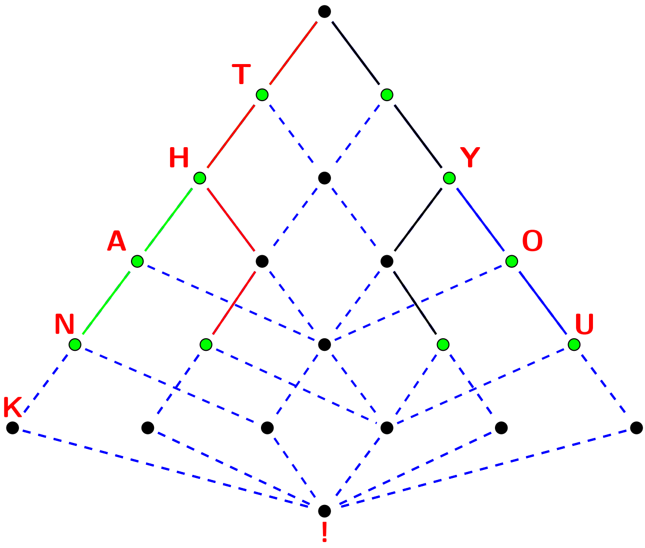
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Sketch of Theorem 1's proof

- Lattice rep. of a choice rule
- Definitions
- Main observations
- Construction of the priorities

Lattice rep. of a choice rule

Example

Let $A = \{1, 2, 3, 4, 5\}$ and C be the 2-capacity-filling choice rule that is collected maximal of $(\succ_\alpha, \succ_\beta)$

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lattice rep. of the ex.

$\{12\mathbf{3}4\mathbf{5}\}$

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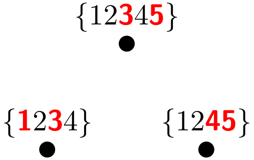

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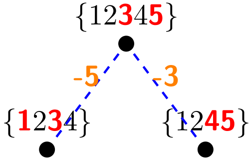

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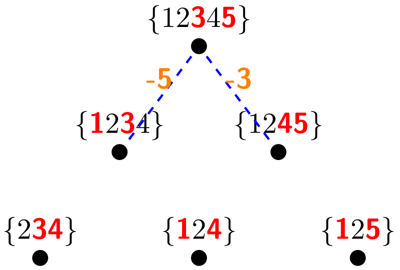

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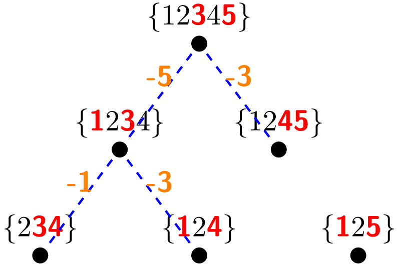

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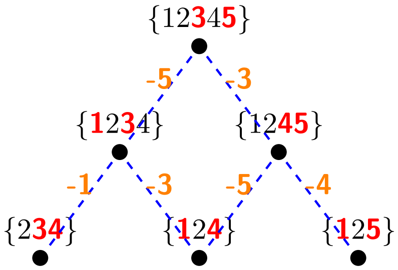

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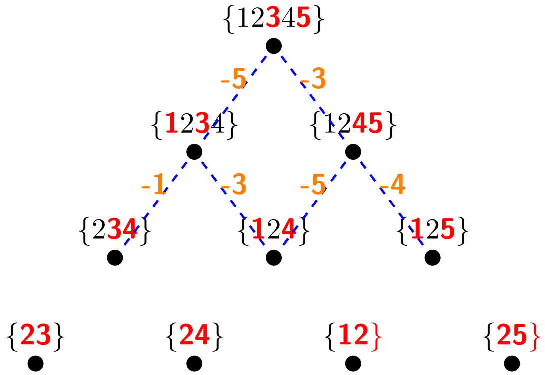

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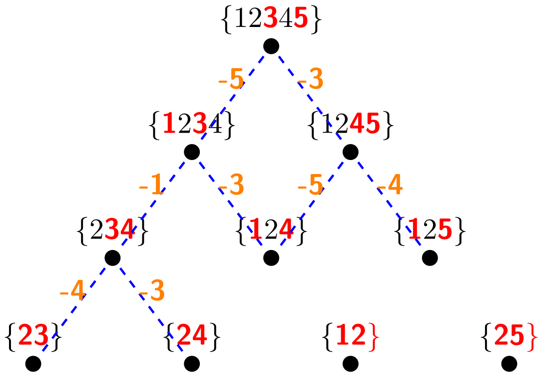

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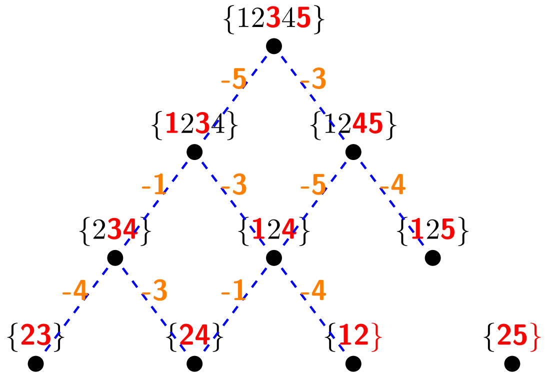

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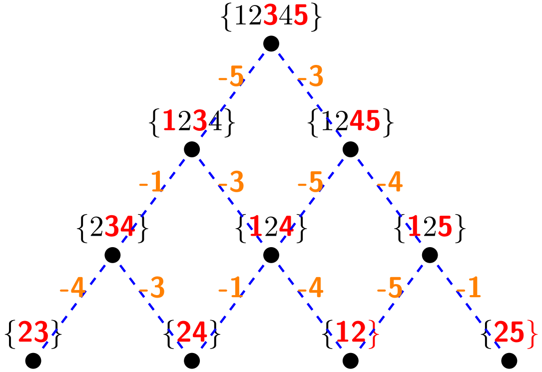

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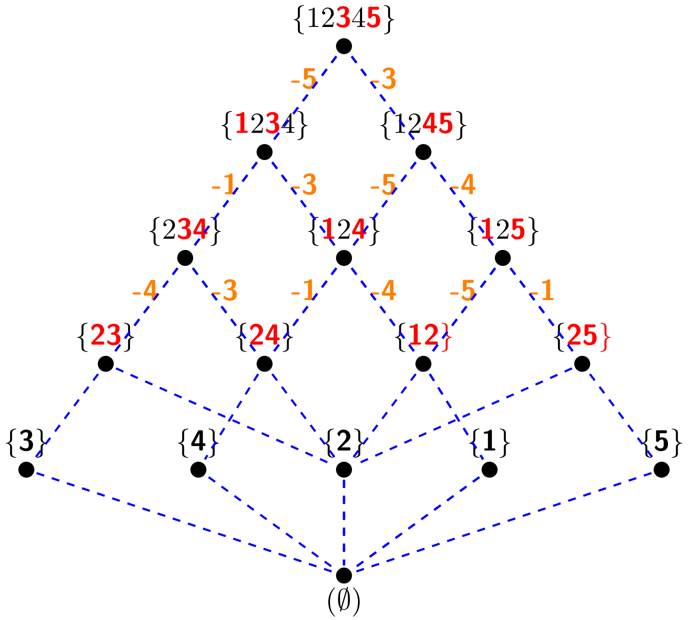

Definitions

maximal choice sets

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$\{\mathbf{1}2\mathbf{3}4\}$ $\{12\mathbf{45}\}$

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S is a parent of S'

$\{12345\}$

$\{1234\}$ $\{1245\}$

$\{234\}$ $\{124\}$ $\{125\}$

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$\forall S, S' \in \mathcal{M}$, S is a parent of S', denoted by $S \to S'$,
if $\exists a \in C(S)$ s.t. $S' = S \setminus \{a\}$.

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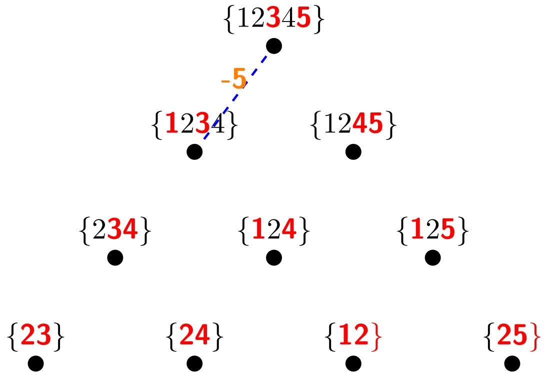

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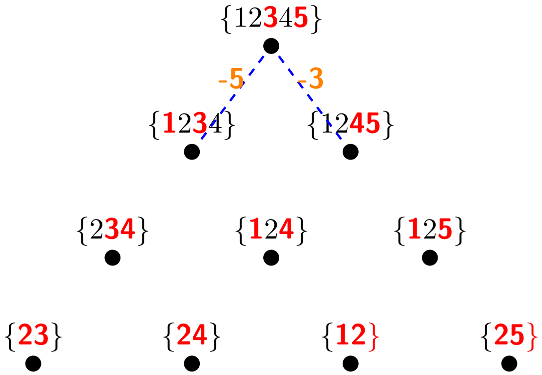

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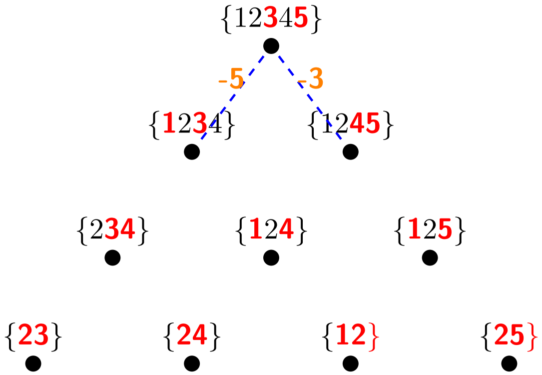

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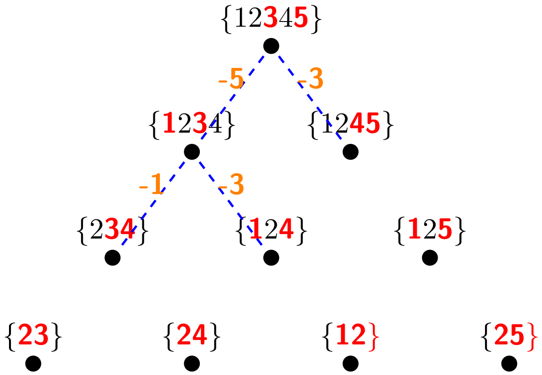

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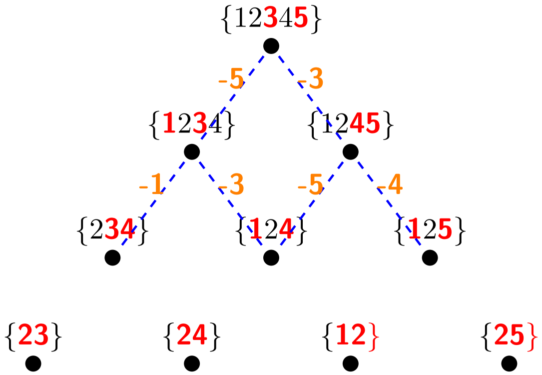

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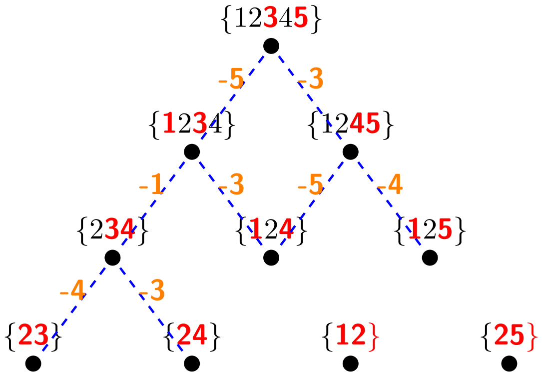

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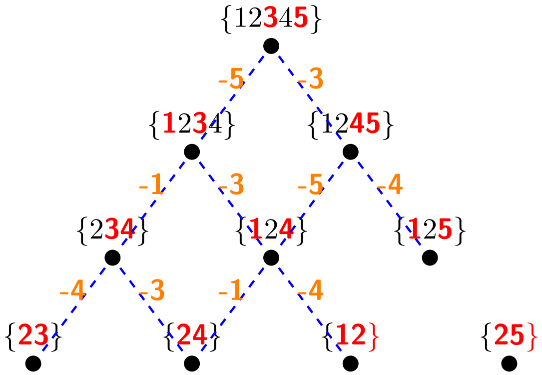

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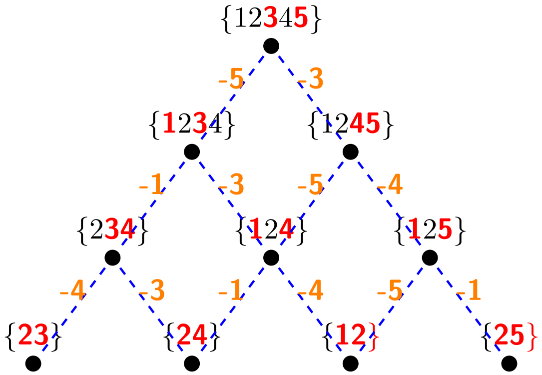

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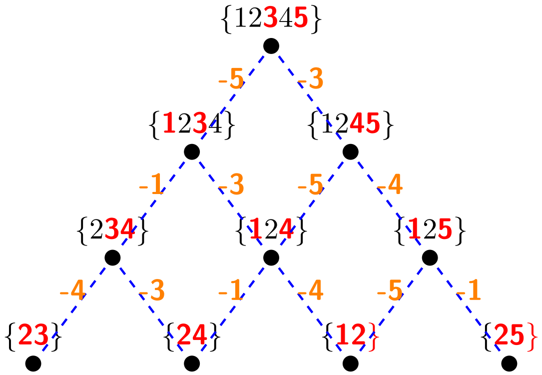

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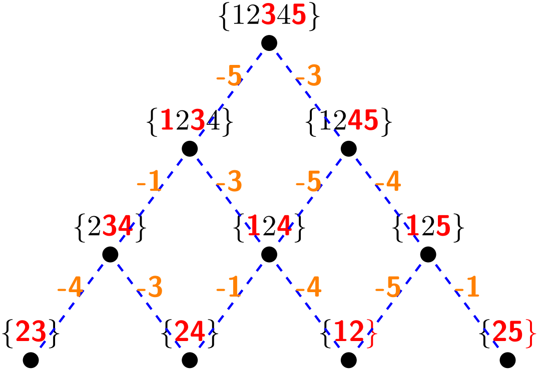
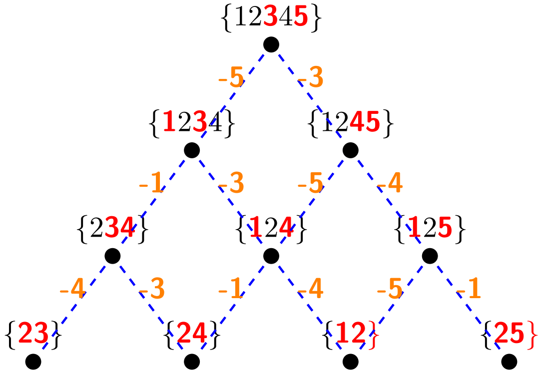


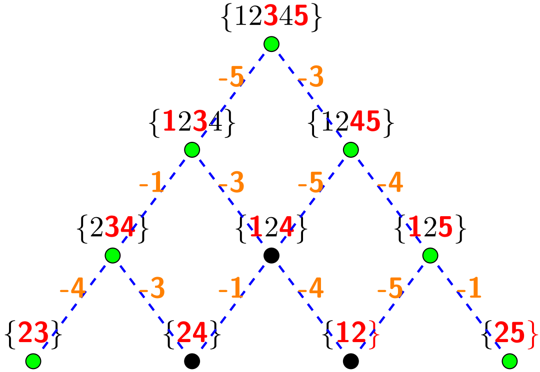
S is an ancestor of S', denoted by $S \searrow S'$, if $S \to S_1 \to \cdots \to S_k \to S'$. Since $\searrow$ is transitive, $(\mathcal{M}, \searrow)$ is a poset.

a chain



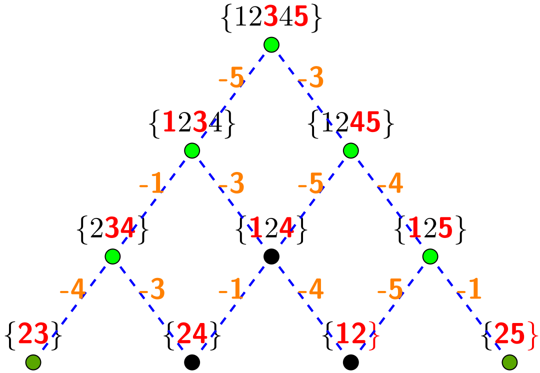
a chain in $(\mathcal{M}, \searrow)$ is a linearly ordered subset of $\mathcal{M}$.

prime sets



$S \in \mathcal{M}$ is a prime if S has a unique parent, i.e., $\exists$ a unique $S' \in \mathcal{M}$ s.t. $S' \to S$. Let $\mathcal{P}$ denote the set of all primes.

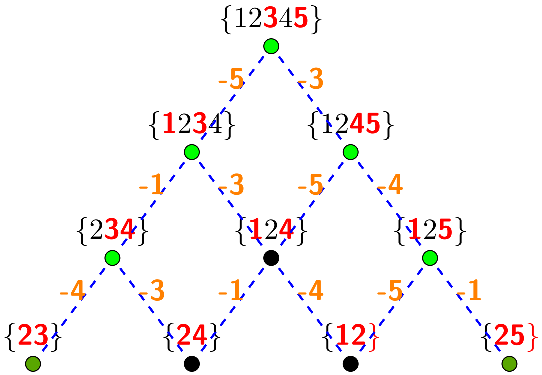
prime chains & prime atoms



$S_1, \ldots, S_k \in \mathcal{P}$ s.t. $S_1 \to \cdots \to S_k$ is a prime chain from S_1 to S_k. Each $T \in \mathcal{M}$ with $|T| = q$ is an atom, if T is also a prime, then T is a prime atom.

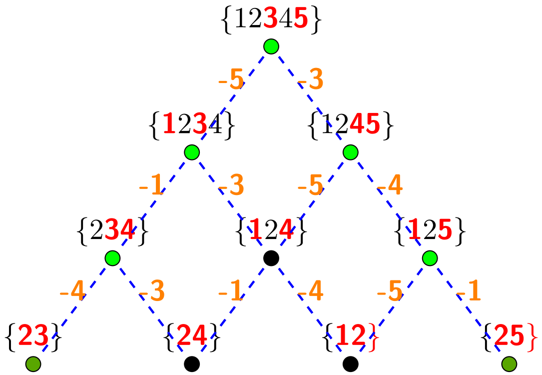
Main observations

prime atoms = critical sets



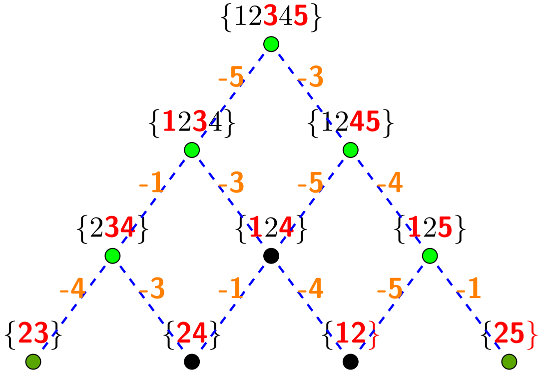
Lemma 1: A choice set is critical if and only if it is a prime atom.

each prime has a prime child



Lemma 2: Each prime choice set has a unique prime child.

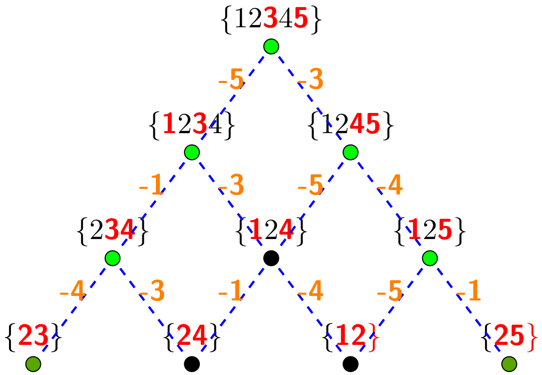
primes carry the whole choice info.



Lemma 3: For each nonprime maximal choice set S and its parent $S \cup \{a\}$, there exists a maximal choice set $S' \in \mathcal{M}$ such that $S \cup \{a\} \subsetneq S'$ and $a \in C(S')$.

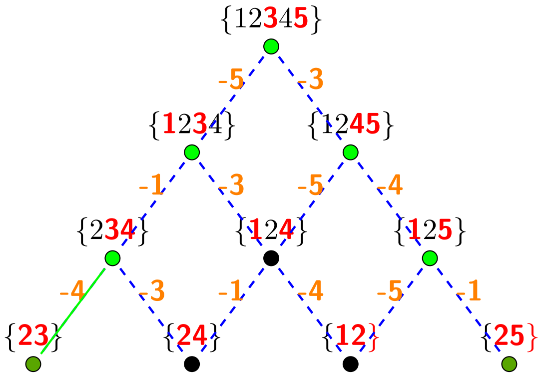
Construction of the priorities

priorities



We construct an ordering associated with each prime atom.

priorities



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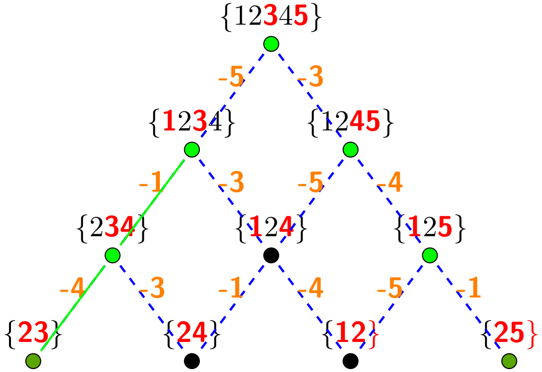
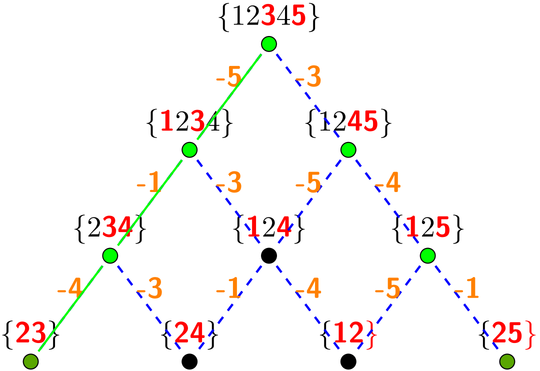


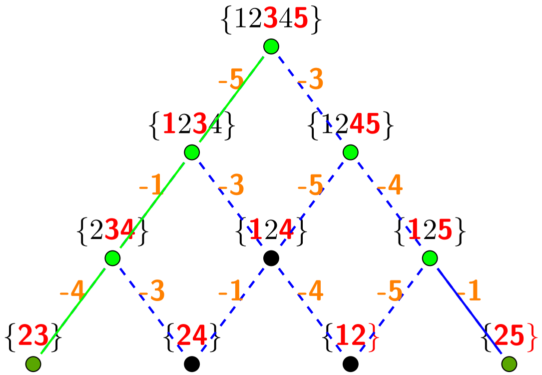
We construct an ordering associated with each prime atom.

priorities



We construct an ordering associated with each prime atom.

priorities



We construct an ordering associated with each prime atom.

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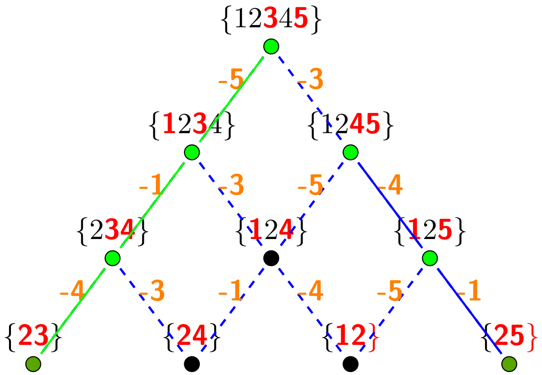
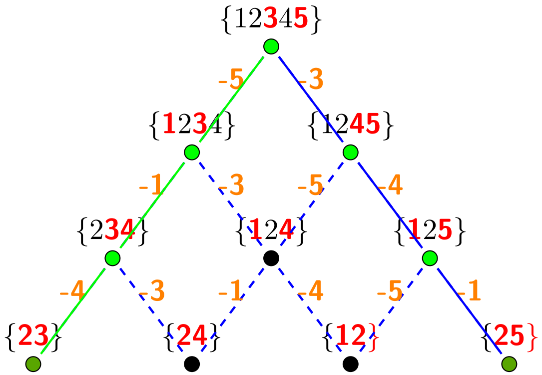


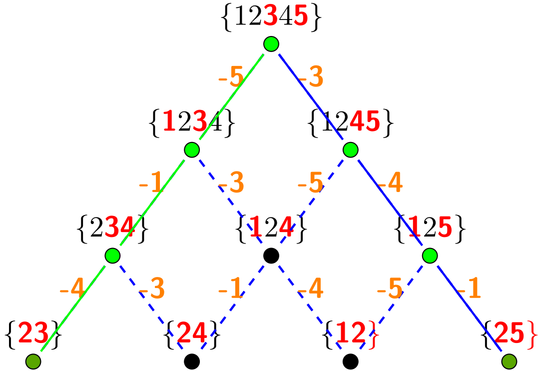
We construct an ordering associated with each prime atom.

priorities



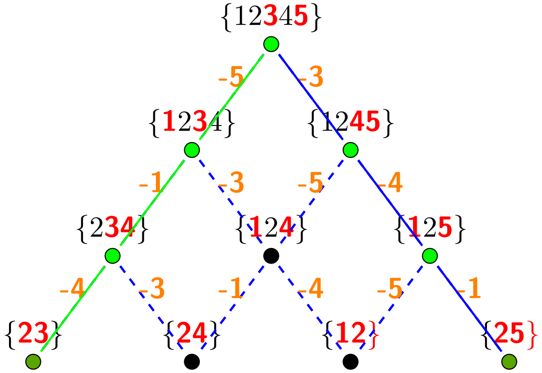
We construct an ordering associated with each prime atom.

priorities



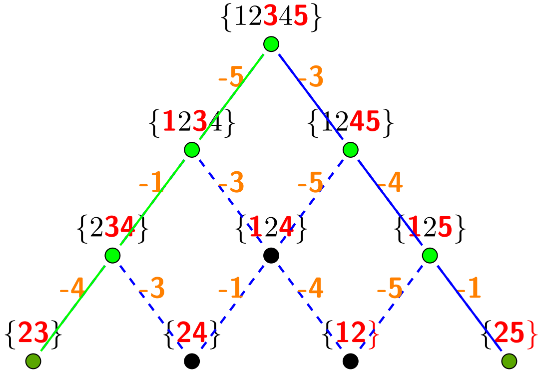
We construct an ordering associated with each prime atom.

priorities



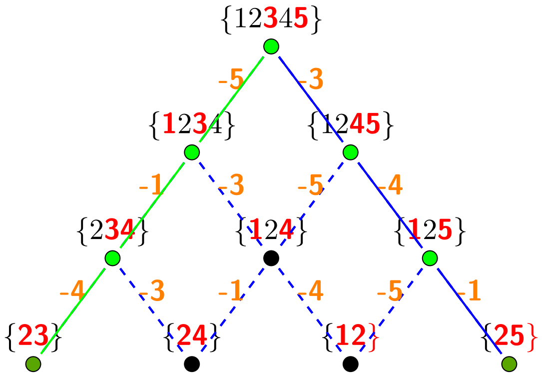
Lemma 3: For each nonprime maximal choice set S and its parent $S \cup \{a\}$, there exists a maximal choice set $S' \in \mathcal{M}$ such that $S \cup \{a\} \subsetneq S'$ and $a \in C(S')$.

priorities



Lemma 3 ⇒ it is sufficient to maximize priorities that passes through the prime sets.

priorities



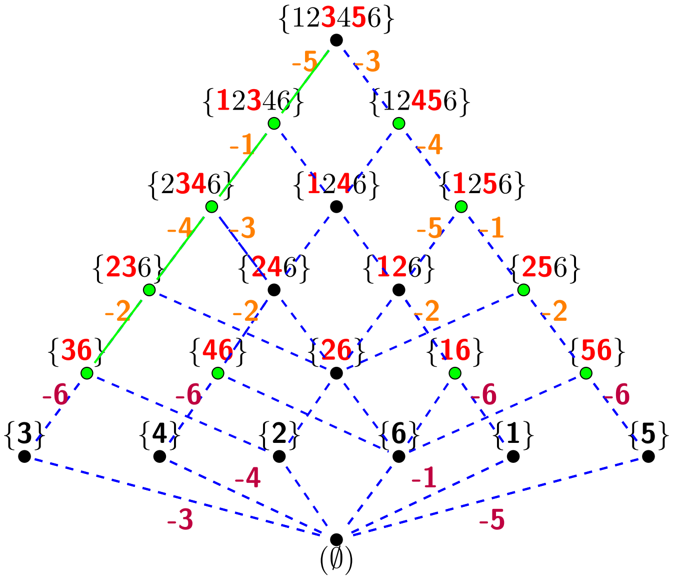
Lemma 2: Each prime choice set has a unique prime child.

resulting priority profile

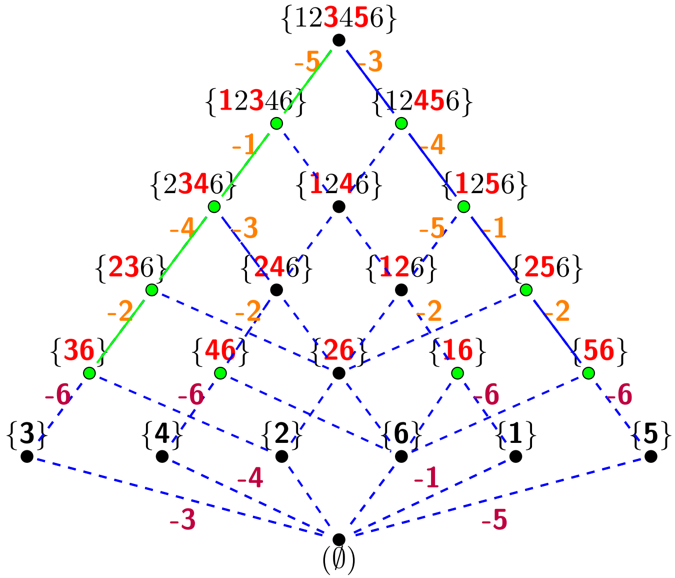
$\succ_\alpha$	$\succ_\beta$
5	3
1	4
4	1
2	2
3	5

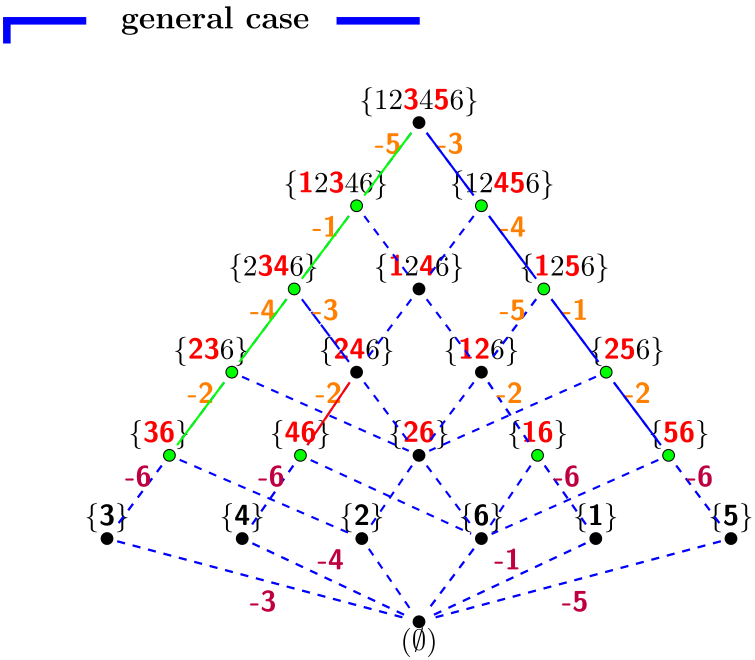
Back

general case

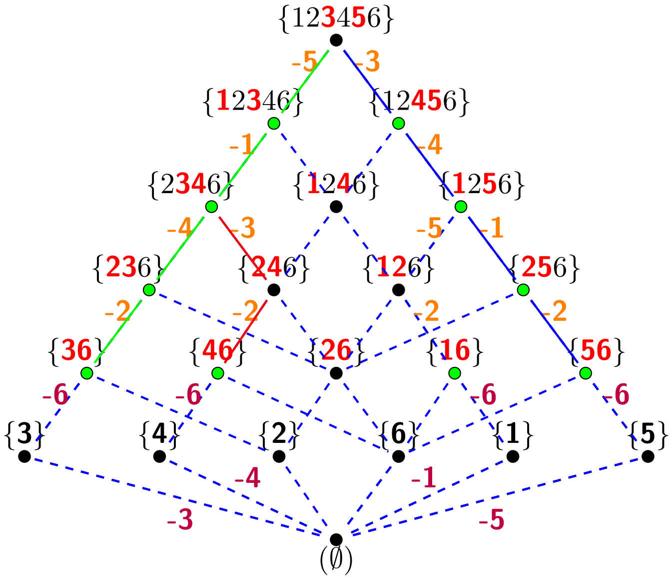


general case

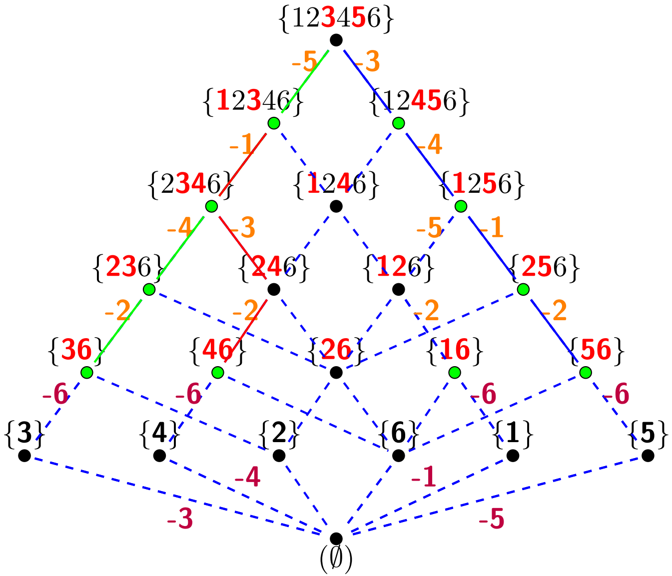



general case
{123456}
-5
-3
{12346}
{12456}
-1
-4
{2346}
{1246}
{1256}
-4
-3
-5
-1
{236}
{246}
{126}
{256}
-2
-2
-2
-2
{36}
{46}
{26}
{16}
{56}
-6
-6
-6
-6
{3}
{4}
{2}
{6}
{1}
{5}
-4
-1
-3
-5
(∅)

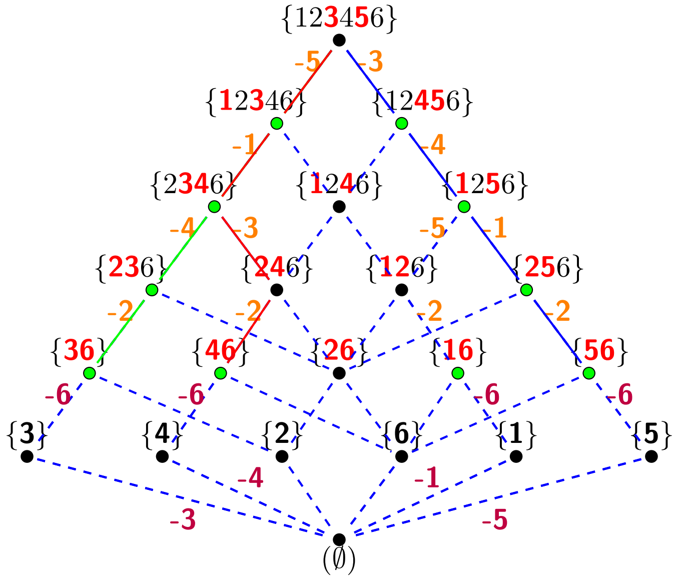
general case



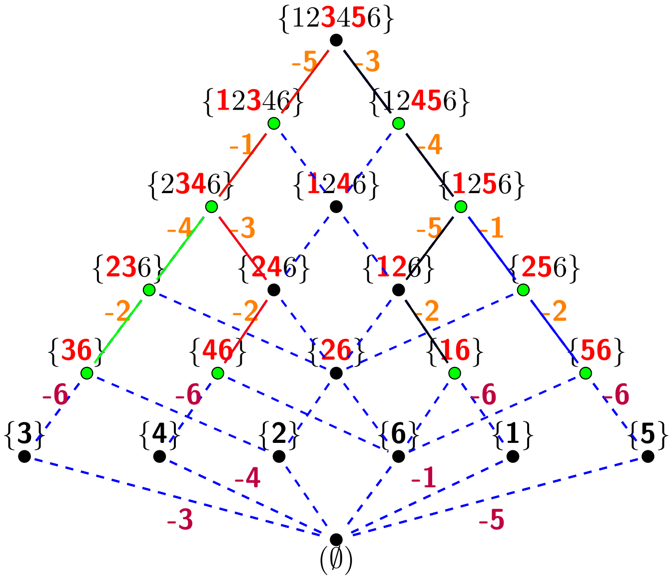
general case



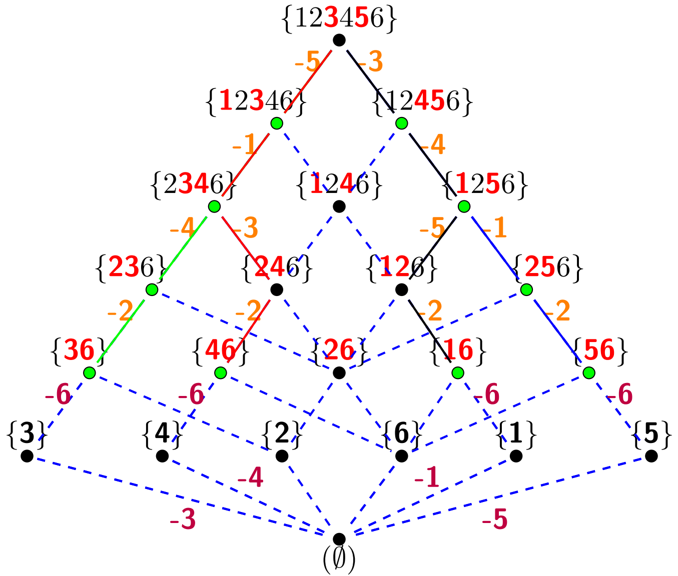
general case



general case



general case



resulting priorities

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
5	5	3	3
1	1	4	4
4	3	5	1
2	2	2	2
6	6	6	6
3	4	1	5

Back

resulting priorities

$\succ_\alpha$	$\succ_\beta$	$\succ_\gamma$	$\succ_\delta$
5	5	3	3
1	1	4	4
4	3	5	1
2	2	2	2
6	6	6	6
3	4	1	5

Back