A new characterization of the Luce rule with an extension to zero probability choices *

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Abstract

We propose a random choice model that extends the Luce rule by allowing *zero probability choices.* We show that a new choice axiom –*rejection supermodularity*, which strengthens the Luce's *regularity* axiom– is the key for relating the model to the observables. A new characterization of the Luce rule follows as a corollary.

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1 Introduction

Luce rule –the most widely used random choice model in economics–asserts that each alternative has a fixed positive weight, and is chosen from a choice set proportion-

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ately to its relative weight. Since the assigned weights are positive, each alternative is chosen from each choice set with a positive probability. On the other hand, for an agent or a group of agents, an alternative may clearly 'dominate' another. Therefore, if both alternatives are available, then the dominated one may never be chosen. The widely adopted formulation of Luce rule[1] discards this possibility, which leads to documented problems in estimating market outcomes.

The common empirical method to deal with zero probability choices is to drop such sample points, or replace them with small numbers. In the international trade literature, Helpman et al. (2008) point out that most of the bias in the traditional estimates is due to the omission of the *extensive margin*, which refers to the exporters that never trade with each other. As for the empirical literature in which Luce rule and its variants are widely used, problems emanating from ignoring zero probability choices have recently received attention. Both Gandhi et al. (2013) and Hortacsu & Joo (2016) argue that when samples with zero market shares are dropped or replaced by small shares, price coefficient estimates are either biased upward or the direction of the bias becomes unpredictable. Hortacsu and Joo demonstrate that ignoring zero choice probabilities can even result in upward-sloping demand curves.[2]

Motivated by these observations, we propose a choice model that extends the Luce rule as to allow zero probability choices by retaining its simplicity. Our theoretical model turns out to be closely related to the empirical model applied by Hortacsu & Joo (2016) to accommodate zero market shares consistently. In our analysis, we show that a new choice axiom, *rejection supermodularity*, which strengthens the Luce's *regularity* axiom, is key for relating the model to the observables. As a corollary, we obtain a new characterization of the Luce rule.

[1] In the original formulation by Luce (1959), it is allowed that some alternatives are never chosen from any choice set. However, it is not allowed that an alternative may be chosen in a choice set but not in another, which is our focus in here. For more details see the recent paper by Horan (2018), who also extends the Luce's *connected domain* approach to accommodate zero choice probabilities.

[2] Hortacsu & Joo (2016) illustrate this by using Dominick's supermarket cola sales scanner data. This data covers 100 chain stores in the Chicago area for 400 weeks, from September 1989 to May 1997.

We formulate the *preference-oriented Luce rule* (POLR), which extends the Luce rule by taking into account the possibility that an alternative may clearly *dominate* another. This model has two primitives: a *preference relation*[3] $\succsim$ that allows for indifferences, and a *weight function* v that assigns a positive real number to each alternative. From each choice set S, an agent first shortlists the $\succsim$-best alternatives in S, then chooses each shortlisted alternative with a probability that equals the alternative's relative weight in the shortlist. That is, the probability of choosing an alternative x from a choice set S is

$$\rho_x(S) = \begin{cases} \dfrac{v(x)}{\displaystyle\sum_{y \in max(S,\succsim)} v(y)} & \text{if } x \in max(S,\succsim) \\ 0 & \text{otherwise.} \end{cases}$$

In our Theorem 1, we show that the preference-oriented Luce rules are the only *random choice functions* (RCFs) that satisfy *rejection supermodularity*. To introduce this new axiom, suppose that an alternative x is chosen from a choice set S with positive probability, the *rejection likelihood of* x *in* S, denoted by $r(x,S)$, is the ratio of the probability that x is not chosen (rejected) from S to the probability that x is chosen from S, that is

$$r(x,S) = \frac{1-\rho_x(S)}{\rho_x(S)}.$$

If x is not chosen from a choice set S with positive probability, then $r(x,S) = \infty$. For each pair of choice sets S and T such that the intersection of S and T only contains x, *rejection supermodularity* requires the sum of the rejection likelihoods of x in S and T be less than or equal to the rejection likelihood of x in $S \cup T$. That is,

$$r(x,S) + r(x,T) \leq r(x,S \cup T).$^{[4]}$$

Rejection supermodularity can be thought as a strengthening of the Luce's *regularity* axiom. *Regularity* requires that if new alternatives are added to a choice set, then the choice probability of an existing alternative x should not increase, equivalently,

[3] A complete and transitive binary relation on the finite alternative set.

[4] This is equivalent to require that $r(x,\cdot)$ is supermodular for each alternative x, that is for each pair of choice sets S and T, $r(x,S) + r(x,T) \leq r(x,S \cup T) + r(x,S \cap T)$.

rejection likelihood of x should not decrease. *Rejection supermodularity* strengthens *regularity* by requiring that the rejection likelihood of x increases at least additively as new alternatives are added to the choice set. The main difference between a preference-oriented Luce rule and the classic Luce rule is *positivity*, which requires each alternative be chosen from each choice set with positive probability. Thus, we obtain a new characterization of the Luce rule in terms of *positivity* and *rejection supermodularity*.

2 Preference-oriented Luce rule

2.1 The model

Given an alternative set X, any nonempty subset S is called a **choice set**. Let Ω denote the collection of all choice sets. A **random choice function** (RCF) p assigns each choice set $S \in \Omega$ a probability measure over S. We denote by $\rho_x(S)$ the probability that alternative x is chosen from choice set S. A preference relation $\succsim$ is a complete and transitive binary relation on X. We denote the strict part of $\succsim$ with $\succ$, and the indifference part of $\succsim$ with $\sim$. Next, we define the preference-oriented Luce rule.

Definition 1 An RCF ρ is a *preference-oriented Luce rule* (POLR) if there is a preference relation $\succsim$ on X and a weight function $v : X \to R_{++}$ such that for each choice set $S \in \Omega$,

$$\rho_x(S) = \begin{cases} \dfrac{v(x)}{\sum\limits_{y \in max(S,\succsim)} v(y)} & \text{if } x \in max(S,\succsim) \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Characterization

Let ρ be an RCF. For each $S \in \Omega$ and $x \in S$, the **rejection likelihood of x in S**, denoted by $r(x,S)$, is the ratio of the probability that x is not chosen from S to the probability that x is chosen from S, i.e. $r(x,S) = \frac{1-\rho_x(S)}{\rho_x(S)}$ if $\rho_x(S) > 0$, and $r(x,S) = \infty$

if $\rho_x(S) = 0$. Next, we introduce our new axiom.

Rejection supermodularity: For each $S, T \in \Omega$ and $x \in X$ such that $S \cap T = \{x\}$,

$$r(x, S) + r(x, T) \leq r(x, S \cup T).$$

It is easy to note that *rejection supermodularity* is equivalent to require that for each $x \in X$, $r(x, \cdot)$ is supermodular, i.e. for each $S, T \in \Omega$, $r(x, S) + r(x, T) \leq r(x, S \cup T) + r(x, S \cap T)$. As we discuss in detail in the introduction, *rejection supermodularity* can be thought as a strengthening of the *regularity* axiom. Next, we present our characterization of preference-oriented Luce rule. We present the proof in Section 4.

Theorem 1 *An RCF ρ is a preference-oriented Luce rule if and only if ρ satisfies rejection supermodularity.*

In the representation, the preference relation is identified uniquely, and the identified weight function is unique up to multiplication by a positive scalar for each equivalence class of the preference relation. In their empirical model, Hortacsu & Joo (2016) consider a consumer who assigns each product a (Luce) weight in a specific way, model the consumer's choice as a two-stage procedure. First, the consumer eliminates the products with weights lower than a given threshold, and then decides how much to consume of each remaining product according to its relative weight. The Luce weights are used both to determine which products are consumed with positive probability, and the market share of each product. If an RCF is a POLR, then we can rescale the weight function and create threshold values for the choice sets so that POLR representation also fits into the form of Hortacsu & Joo (2016)'s empirical model.[5]

The main difference between a preference-oriented Luce rule and the classic Luce rule is *positivity*, which requires each alternative be chosen from each choice set with positive probability, i.e. for each $S \in \Omega$ and $x \in S$, $\rho_x(S) > 0$. As a corollary to

[5] In Hortacsu & Joo (2016)'s formulation, the threshold value is fixed, yet the weight of an alternative has a choice set dependent component. In mapping a POLR to a threshold model, we keep the weights fixed, but allow the threshold values depend on the choice sets such that as new alternatives are added to a choice set the threshold value does not decrease.

Theorem 1, we obtain a new characterization of the Luce rule in in terms of *positivity* and *rejection supermodularity*.

Corollary 1 *An RCF ρ is a Luce rule if and only if ρ satisfies positivity and rejection supermodularity.*

Luce (1959) characterizes his model in terms of *Luce's IIA*, which requires that for each pair of alternatives x and y and choice set S that contains x and y, $\rho_x(\{x, y\})/\rho_y(\{x, y\}) = \rho_x(S)/\rho_y(S)$. While they both characterize the Luce rule together with *positivity*, IIA and *rejection supermodularity* have substantially different formulations. Our Lemma 5 and Lemma 6 show the connection between the two.

3 Relation to the literature and final comments

We analyzed a structured extension of the Luce rule that accommodates zero probability choices. Hortacsu & Joo (2016) develop an empirical choice model to accommodate zero choice probabilities. As we highlight in Section 2.2, by rescaling the Luce weights and choosing the thresholds properly, our POLR can also be represented in this form. Thus, besides its normative appeal, our characterization of POLR paves the way for identification of a model that is empirically relevant.

Echenique & Saito (2015) and Ahumada & Ulku (2017) analyze a general Luce model that accommodates zero probability choices. In this model, an agent first forms his consideration sets in any arbitrary way, then applies the Luce rule. Our POLR can be seen as a general Luce rule with specific consideration set structure. A key difference between our work and these studies is our axiomatic characterization. In that, *rejection supermodularity* plays the main role in our results, whereas *cyclical independence*, which is a multiplicative extension of *Luce's IIA*, plays the main role in theirs. Yildiz (2013) analyzes a specific POLR, in which the preference to be maximized is a weak order such that an alternative can be indifferent with at most one other alternative. He shows that this model is characterized by a random counterpart of Plott (1973)'s *path independence*. POLR generalizes this model by allowing an agent to be indifferent among any number of alternatives.

We obtain a new axiomatic characterization of the Luce rule in terms of *positivity* and *rejection supermodularity*. Apart from Luce's original characterization in terms of IIA, ours seems to be the first alternative characterization of the Luce rule in the in its classical setting. There are two recent papers that offer new axiomatizations of the Luce rule in more structured choice environments. Gul et al. (2014) offer a characterization of the Luce rule for the "rich" choice environment. An implication of *richness* is that the alternative set should be infinite. Ahn et al. (2018) characterize the counterpart of the Luce rule in "average choice" environment, in which the alternative set is also infinite and endowed with a vector space structure that allows for the existence of convex combinations and averages. Here, we retain the original setting of Luce (1959), and consider a finite set of alternatives. These different assumptions on the choice environments lead to different characterizations of the Luce rule. The three main axioms, namely our *rejection supermodularity*, Gul et al.'s *independence*, and Ahn et al's *partial path independence* have substantially different formulations that shed light on different aspects of the Luce rule. We believe that our characterization of the Luce rule in terms of *rejection supermodularity* illuminates a connection that goes unnoticed, and might be fruitful in analyzing the extensions of the Luce rule in its classical setting.

4 Proof of Theorem 1

For any given RCF ρ, define the binary relation $\succsim_\rho$ on X such that for each $x, y \in X$, $x \succsim_\rho y$ if and only if $\rho_x(\{x, y\}) > 0$. It follows that $x \succ_\rho y$ if and only if $\rho_x(\{x, y\}) = 1$. In the sufficiency part of the proof, we show that the $\succsim_\rho$ relation is the relation that is to be maximized. In Lemma 3, we show that if an RCF ρ satisfies *rejection supermodularity*, then $\succsim_\rho$ is complete and transitive. To prove this result, first we prove the following two lemmas. Lemma 1 shows that if ρ satisfies *rejection supermodularity*, then ρ satisfies *regularity*, i.e. for each $S, T \in \Omega$ such that $S \subset T$, and $x \in S$, $\rho_x(T) \leq \rho_x(S)$.

Lemma 1 *If an RCF ρ satisfies rejection supermodularity, then ρ satisfies regularity.*

Proof. For each $T \in \Omega$ and $x \in T$ with $\rho_x(T) > 0$, $r(x, T) < \infty$. Since for each $S \subset T$ such that $x \in S$, $r(x, S) + r(x, (T \setminus S) \cup \{x\}) \leq r(x, T)$, we have $r(x, S) \leq r(x, T)$. It follows that $\rho_x(T) \leq \rho_x(S)$. ■

Lemma 2 *Let ρ be an RCF that satisfies rejection supermodularity. For each $S \in \Omega$ and $x, y \in S$, if $\rho_x(S) > 0$ and $\rho_y(S) = 0$, then $\rho_x(\{x, y\}) = 1$.*

Proof. For each $S \in \Omega$ and $x, y \in S$, since ρ satisfies *rejection supermodularity*, we have $r(x, \{x, y\}) + r(x, S \setminus \{y\}) \leq r(x, S)$. Suppose $\rho_x(S) > 0$ and $\rho_y(S) = 0$, we show that $\rho_x(\{x, y\}) = 1$. We first show that $\rho_x(S \setminus \{y\}) = \rho_x(S)$. To see this, note that since ρ satisfies *regularity* by Lemma 1, for each $z \in S \setminus \{y\}$, $\rho_z(S \setminus \{y\}) \geq \rho_z(S)$. Suppose $\rho_x(S \setminus \{y\}) > \rho_x(S)$. Since the choice probabilities sum to one and $\rho_y(S) = 0$, there exists $z \in S \setminus \{x, y\}$ with $\rho_z(S \setminus \{y\}) < \rho_z(S)$. This is a contradiction. Now, since $\rho_x(S \setminus \{y\}) = \rho_x(S)$, we have $r(x, S \setminus \{y\}) = r(x, S)$. Therefore, $r(x, \{x, y\}) = 0$, which implies $\rho_x(\{x, y\}) = 1$. ■

Lemma 3 *If an RCF ρ satisfies rejection supermodularity, then $\succsim_\rho$ is complete and transitive.*

Proof. Since either $\rho_x(\{x, y\}) \geq 0$ or $\rho_y(\{x, y\}) \geq 0$, $\succsim_\rho$ is complete. To see that $\succsim_\rho$ is transitive, by contradiction suppose there exist $x, y, z \in X$ such that $x \succsim_\rho y \succsim_\rho z$ and $z \succ_\rho x$, which means $\rho_x(\{x, z\}) = 0$. Now, consider $S = \{x, y, z\}$, since ρ satisfies *regularity* and $\rho_x(\{x, z\}) = 0$, $\rho_x(S) = 0$. Since $\rho_x(S) = 0$ and $x \succsim_\rho y$, it follows from Lemma 2 that $\rho_y(S) = 0$. Therefore, $\rho_z(S) = 1$. But, since ρ satisfies *regularity*, we must have $\rho_z(\{y, z\}) = 1$, which contradicts to $y \succsim_\rho z$. ■

Lemma 4 *Let ρ be an RCF that satisfies rejection supermodularity. For each $S \in \Omega$ and $x \in S$, $\rho_x(S) > 0$ if and only if $x \in \max(S, \succsim_\rho)$.*

Proof. If part: For each $S \in \Omega$ and $x \in S$, $x \in \max(S, \succsim_\rho)$ implies for each $y \in S \setminus \{x\}$, $\rho_x(\{x, y\}) \geq 0$. Since $\rho_y(S) > 0$ for some $y \in S$, it follows from Lemma 2 that $\rho_x(S) > 0$. Only if part: Suppose $\rho_x(S) > 0$, we show that $x \in \max(S, \succsim_\rho)$. By contradiction, suppose there exists $y \in S$ such that $y \succ_\rho x$. It follows that $\rho_x(\{x,y\}) = 0$. This together with the regularity of ρ implies $\rho_x(S) = 0$. This contradicts that $\rho_x(S) > 0$.

■

Next we present two critical results for the construction of the Luce weights.

Lemma 5 *For each* $x, y, z \in R_{++}$, $\frac{x}{1+x+xy} + \frac{y}{1+y+yz} + \frac{z}{1+z+zx} \leq 1$. *Moreover, equality holds if and only if* $xyz = 1$.[6]

Proof. If we substitute $\frac{1}{1+x+xy} + \frac{x}{1+x+xy} + \frac{xy}{1+x+xy}$ instead of 1, then we get the following inequality

$$\frac{x}{1+x+xy} + \frac{y}{1+y+yz} + \frac{z}{1+z+zx} \overset{?}{\leq} \frac{1}{1+x+xy} + \frac{x}{1+x+xy} + \frac{xy}{1+x+xy}.$$

By rearranging the terms we get

$$\left[\frac{y}{1+y+yz} - \frac{xy}{1+x+xy}\right] + \left[\frac{z}{1+z+zx} - \frac{1}{1+x+xy}\right] \overset{?}{\leq} 0.$$

By equating denominators we get

$$\frac{y + xy + xy^2 - xy - xy^2 - xy^2 z}{(1+y+yz)(1+x+xy)} + \frac{z + zx + xyz - 1 - z - zx}{(1+z+zx)(1+x+xy)} \overset{?}{\leq} 0.$$

Then,

$$\frac{y(1-xyz)}{(1+y+yz)(1+x+xy)} - \frac{1-xyz}{(1+z+zx)(1+x+xy)} \overset{?}{\leq} 0.$$

Finally we obtain

$$\frac{1-xyz}{1+x+xy}\left(\frac{xyz-1}{(1+y+yz)(1+z+zx)}\right) = \frac{-(1-xyz)^2}{(1+x+xy)(1+y+yz)(1+z+zx)} \overset{?}{\leq} 0.$$

It is clear that this inequality holds, and it holds as an equality if and only if $xyz = 1$.

■

Lemma 6 *Let* ρ *be an RCF that satisfies rejection supermodularity. For each distinct* $x, y, z \in X$, *if* $\rho_x(x,y) \in (0,1)$, $\rho_x(x,z) \in (0,1)$, *and* $\rho_y(y,z) \in (0,1)$, *then* $\frac{r(x,\{x,z\})}{r(x,\{x,y\})} = \frac{\rho_z(\{y,z\})}{\rho_y(\{y,z\})}$.

[6] Selim Bahadir from the Mathematics department of Yildirim Beyazit University, Ankara, Turkey proved the extension of this result for more than three positive numbers.

Proof. Consider the choice set $\{x, y, z\}$. Since for each $S, T \in \Omega$ with $S \cap T = \{x\}$, we have $\rho_x(S \cup T) = \dfrac{1}{1 + r(x, S \cup T)}$. Then, *rejection supermodularity* implies that $\rho_x(S \cup T) \leq \frac{1}{1+r(x,S)+r(x,T)}$. It follows that

$$\rho_x(x,y,z) \leq \frac{1}{r(x,\{x,z\}) + 1 + r(x,\{x,y\})} = \frac{r(z,\{x,z\})}{1 + r(z,\{x,z\}) + r(z,\{x,z\})r(x,\{x,y\})},$$
$$\rho_y(x,y,z) \leq \frac{1}{r(y,\{x,y\}) + 1 + r(y,\{y,z\})} = \frac{r(x,\{x,y\})}{1 + r(x,\{x,y\}) + r(x,\{x,y\})r(y,\{y,z\})},$$
$$\rho_z(x,y,z) \leq \frac{1}{r(z,\{y,z\}) + 1 + r(z,\{x,z\})} = \frac{r(y,\{y,z\})}{1 + r(y,\{y,z\}) + r(y,\{y,z\})r(z,\{x,z\})}.$$

Note that to obtain the rightmost terms, we multiply both the numerator and the denominator of the middle terms with $r(z, \{x, z\})$, $r(x, \{x, y\})$, and $r(y, \{y, z\})$ respectively. Now, consider the sum of the three rightmost terms. It follows from Lemma 5 that this sum is less than or equal to 1. On the other hand, since the sum of the three leftmost terms equals 1, the sum of the rightmost terms must be equal to 1. By Lemma 5, we know that this equality holds if and only if $r(x, \{x, y\}) \cdot r(y, \{y, z\}) \cdot r(z, \{x, z\}) = 1$. If we substitute $\frac{1}{r(x,\{x,z\})}$ instead of $r(z, \{x, z\})$, then we get $r(y, \{y, z\}) = \frac{r(x,\{x,z\})}{r(x,\{x,y\})}$. Since $r(y, \{y, z\}) = \frac{\rho_z(\{y,z\})}{\rho_y(\{y,z\})}$, we get the desired conclusion. ■

Now, we proceed to prove Theorem 1.

Only if part: Let ρ be a preference-oriented Luce rule represented by $\succsim$ and v. First, suppose $r(x, S \cup T) = \infty$ for some $S, T \in \Omega$ such that $S \cap T = x$. It follows that $\rho_x(S \cup T) = 0$. This implies there exists $y \in S \cup T$ such that $y \succ x$. It follows that either $y \in S$ or $y \in T$. Therefore, either $r(x, S) = \infty$ or $r(x, T) = \infty$. Thus, we obtain $r(x, S) + r(x, T) = r(x, S \cup T)$. Next, suppose $r(x, S \cup T) < \infty$. It follows that $\rho_x(S \cup T) > 0$ and $x \in \max(S \cup T, \succsim)$. Let $S' = \max(S, \succsim) \setminus \{x\}$ and $T' = \max(T, \succsim) \setminus \{x\}$. Since $x \in max(S, \succsim)$ and $x \in max(T, \succsim)$, all the alternatives in S' and T' belong to the same indifference class of $\succsim$. It follows that $\max(S \cup T, \succsim) = S' \cup T' \cup \{x\}$. Now, since ρ is a preference-oriented Luce rule, we have

$$\rho_x(S) = \frac{v(x)}{v(x) + \sum\limits_{y \in S'} v(y)}, \ \rho_x(T) = \frac{v(x)}{v(x) + \sum\limits_{y \in T'} v(y)}, \text{ and } \rho_x(S \cup T) = \frac{v(x)}{v(x) + \sum\limits_{y \in S' \cup T'} v(y)}$$

It follows that

$$r(x,S) = \frac{\sum\limits_{y\in S'} v(y)}{v(x)}, \quad r(x,T) = \frac{\sum\limits_{y\in T'} v(y)}{v(x)}, \text{ and } \quad r(x,S\cup T) = \frac{\sum\limits_{y\in S'\cup T'} v(y)}{v(x)}$$

Since $S \cap T = \{x\}$, we have $S' \cap T' = \emptyset$. Thus, we obtain that $r(x,S) + r(x,T) = r(x, S\cup T)$.

If part: Suppose ρ is an RCF that satisfies *rejection supermodularity*, we construct a weak order $\succsim$ and a weight function $v : X \to R_+$ that recovers the choices of ρ. We choose $\succsim$ as the $\succsim_\rho$ that is defined above. It follows from Lemma 3 that $\succsim_\rho$ is a weak order. It follows from Lemma 4 that for each $S \in \Omega$ and $x \in S$, $\rho_x(S) > 0$ if and only if $x \in \max(S, \succsim_\rho)$.

Next, we construct the weight function $v : X \to R_+$. First for each $x \in X$, define $x_\sim = \{z \in X \;\; x \sim_\rho z\}$. Next, we specify the weights for each of these equivalence classes. For each $x \in X$, if $x_\sim = \{x\}$, i.e. there is no $z \in X \setminus \{x\}$ with $z \sim_\rho x$, then let $v(x) = 1$. If there is a single $z \in X \setminus \{x\}$ with $z \sim_\rho x$, then let $v(x) = \rho_x(\{x,z\})$ and $v(z) = \rho_z(\{x,z\})$. If $|x_\sim| > 2$, then define $v(x) = 1$ for some fixed $x \in x_\sim$, and for each $z \in x_\sim \setminus \{x\}$, define $v(z) = r(x, \{x,z\})$. Thus we complete the construction of the weight function v.

Now, let $\widehat{\rho}$ be the preference-oriented Luce rule defined by $\succsim_\rho$ and v. First, we show that for each $x, y \in X$ we have $\rho_x(\{x,y\}) = \widehat{\rho}_x(\{x,y\})$. To see this, note that for each $x, y \in X$, if $\rho_x(\{x,y\}) = 1$ ($\rho_x(\{x,y\}) = 0$), then $x \succ_\rho y$ ($y \succ_\rho x$). Therefore, $\widehat{\rho}_x(\{x,y\}) = 1$ ($\widehat{\rho}_x(\{x,y\}) = 0$). Next we show that if $x \sim_\rho y$, then $\frac{v(x)}{v(y)} = \frac{\rho_x(\{x,y\})}{\rho_y(\{x,y\})}$. Once we show this, then it directly follows that $\rho_x(\{x,y\}) = \widehat{\rho}_x(\{x,y\})$. To see this, let $x^* \in x_\sim$ such that $v(x^*) = 1$ and for each $z \in x_\sim$, $v(z) = r(x^*, \{x^*, z\})$. If $x = x^*$ or $y = x^*$, then it directly follows from the definition of rejection likelihood that $\frac{v(x)}{v(y)} = \frac{\rho_x(\{x,y\})}{\rho_y(\{x,y\})}$. If $x^* \notin \{x,y\}$, then it follows from Lemma 6 that $\frac{r(x^*,\{x^*,x\})}{r(x^*,\{x^*,y\})} = \frac{\rho_x(\{x,y\})}{\rho_y(\{x,y\})}$. Since $v(x) = r(x^*, \{x^*, x\})$ and $v(y) = r(x^*, \{x^*, y\})$, we obtain the desired conclusion.

Next, consider each $S \in \Omega$ and $x \in S$, note that by applying *rejection supermodularity* recursively, we get $r(x,S) \geq \sum_{y\in S\setminus\{x\}} r(x, \{x,y\})$. Next, we argue that this inequality should be satisfied as an equality. By contradiction, suppose there exist $S \in \Omega$ and $x^* \in S$ such that $r(x^*, S) > \sum_{y \in S \setminus \{x^*\}} r(x^*, \{x^*, y\})$. First, recall that there exist Luce weights $\{v(x)\}_{x \in X}$ such that for each $x, y \in X$, $r(x, \{x, y\}) = v(y)/v(x)$. Next, multiply these weights with $1/\Sigma_{x \in S} v(x)$, and let v' denote the new Luce weights. We have for each $x, y \in X$, $r(x, \{x, y\}) = v'(y)/v'(x)$. Therefore, for each $x \in S$, $\sum_{y \in S \setminus \{x\}} r(x, \{x, y\}) = \frac{1 - v'(x)}{v'(x)}$. On the other hand, by definition, $r(x, S) = (1 - \rho_x(S))/\rho_x(S)$. It follows that for each $x \in S$ $\rho_x(S) \geq v'(x)$ and $\rho_{x^*}(S) > v'(x^*)$. But, this implies $\sum_{x \in X} \rho_x(S) > \sum_{x \in X} v'(x) = 1$. Thus we get a contradiction. Finally, since for each $S \in \Omega$ and $x \in S$, $r(x, S) = \sum_{y \in S \setminus \{x\}} r(x, \{x, y\})$, choices from pairs of alternatives uniquely specify ρ. Since for each pair of alternatives ρ and $\widehat{\rho}$ assign the same choice probabilities, we get $\rho = \widehat{\rho}$. It follows that ρ is a preference-oriented Luce rule.

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