

A new characterization of the Luce rule ^{*}

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Abstract

We present a characterization of the Luce rule in terms of *positivity* and a new choice axiom called *rejection supermodularity* that strengthens the *regularity*. This new characterization illuminates a connection that goes unnoticed, and sheds light on the behavioral underpinnings of the Luce rule and its extensions from a different perspective. We also show that *rejection supermodularity* per se characterizes a structured extension of the Luce rule that accommodates zero probability choices.

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1 Introduction

Luce rule –the most widely used random choice model in economics–asserts that each alternative has a fixed positive weight, and is chosen from a choice set proportionately to its relative weight. We present a new characterization of the Luce rule in which the classical Luce axioms are substituted by a strengthening of the *regularity* axiom. The classical Luce’s axioms, namely *independence of irrelevant alternatives* (IIA) and the *choice axiom*, play a fundamental role in characterizing choice models that are widely used in empirical work. Examples include axiomatic characterizations of *generalized discounted logit* (Fudenberg and Strzalecki (2015)), *multinomial logit* (Cerrei-Vioglio et al. (2017)), and *mixed logit* (Saito (2017)) models.¹ This new characterization illuminates a connection that goes unnoticed, and sheds light on the behavioral underpinnings of the Luce rule and its extensions from a different perspective.

To introduce our new axiom, consider an alternative x that is chosen from a choice set S with positive probability. The *rejection likelihood of x in S* , denoted by $r(x, S)$, is the ratio of the probability that x is not chosen (rejected) from S to the probability that x is chosen from S , that is

$$r(x, S) = \frac{1 - \rho_x(S)}{\rho_x(S)}.$$

If x is not chosen from a choice set S with positive probability, then $r(x, S) = \infty$. For each pair of choice sets S and T that contain only x in their intersection, *rejection supermodularity* requires the sum of the rejection likelihoods of x in S and T be less than or equal to the rejection likelihood of x in $S \cup T$. That is,

$$r(x, S) + r(x, T) \leq r(x, S \cup T).^2$$

Rejection supermodularity can be interpreted as a strengthening of the *regularity* axiom, which requires that if new alternatives are added to a choice set, then the

¹For the characterizations of these models, our new axiom can replace the Luce’s IIA or the *choice axiom*.

² This is equivalent to require that $r(x, \cdot)$ is supermodular for each alternative x , that is for each pair of choice sets S and T that contain x , $r(x, S) + r(x, T) \leq r(x, S \cup T) + r(x, S \cap T)$.

choice probability of an existing alternative should not increase, equivalently, rejection likelihood of an existing alternative should not decrease. *Rejection supermodularity* strengthens *regularity* by requiring that the rejection likelihood of an existing alternative increases at least additively as new alternatives are added to the choice set. We show that a random choice function is a Luce rule if and only if it satisfies *rejection supermodularity* and *positivity*, which requires each alternative be chosen from each choice set with positive probability.³ After presenting our results, we compare and contrast *rejection supermodularity* to Luce’s original axioms.

We also characterize random choice functions that satisfy *rejection supermodularity*, but possibly fails to satisfy *positivity*. We show that *the preference-oriented Luce rules* (POLRs) are the only random choice functions that satisfy *rejection supermodularity*.⁴ A POLR has two primitives: a *preference relation*⁵ \succsim that allows for indifferences, and a *weight function* v that assigns a positive real number to each alternative. From each choice set S , an agent first shortlists the \succsim -best alternatives in S , then chooses each shortlisted alternative with a probability that equals the alternative’s relative weight in the shortlist. Since POLR allows an alternative to ‘dominate’ another, if both alternatives are available, then the dominated one is never chosen. However, the widely adopted formulation of the Luce rule discards this possibility,⁶ which leads to documented problems due to ‘zero probability choices’ in estimating market outcomes. POLR offers an extension of the Luce rule that allows zero probability choices, yet retain the simplicity of the Luce rule.

³One can easily verify that the Luce rule satisfies the requirement of *rejection supermodularity* as an equality. Since we show that the converse is also true, it follows that whenever a random choice function satisfies *rejection supermodularity* it satisfies in the form of an equality.

⁴A recent study [Cerrei-Vioglio et al. \(2017\)](#) show that *the choice axiom* of Luce also characterizes POLR. It follows that despite their different formulations *rejection supermodularity* and the *choice axiom* can be used interchangeably in characterizations.

⁵A preference relation is a complete and transitive binary relation on the finite alternative set.

⁶In the original formulation by [Luce \(1959\)](#), it is allowed that some alternatives are never chosen from any choice set. However, it is not allowed that an alternative may be chosen in a choice set but not in another, which is our focus in here. For more details see the recent paper by [Horan \(2018\)](#), who also extends the Luce’s *connected domain* approach to accommodate zero choice probabilities.

The common empirical method to deal with zero probability choices is to drop such sample points, or replace them with small numbers.⁷ However, in the international trade literature, [Helpman et al. \(2008\)](#) point out that most of the bias in the traditional estimates is due to the omission of the *extensive margin*, which refers to the exporters that never trade with each other. As for the empirical literature in which Luce rule and its variants are widely used, problems emanating from ignoring zero probability choices have recently received attention. Both [Gandhi et al. \(2013\)](#) and [Hortacsu and Joo \(2016\)](#) argue that when samples with zero market shares are dropped or replaced by small shares, price coefficient estimates are either biased upward or the direction of the bias becomes unpredictable. Hortacsu and Joo demonstrate that ignoring zero choice probabilities can even result in upward-sloping demand curves.⁸ As we discuss in Section 3, POLR turns out to be closely related to the empirical model developed by [Hortacsu and Joo \(2016\)](#) to accommodate zero market shares consistently. In what follows, we first introduce and characterize the POLR. As a corollary of this result, we obtain the new characterization of the Luce rule. We conclude by relating our work to the existing literature.

2 The model

Given an alternative set X , any nonempty subset S is called a **choice set**. Let Ω denote the collection of all choice sets. A **random choice function** (RCF) p assigns each choice set $S \in \Omega$ a probability measure over S . We denote by $\rho_x(S)$ the probability that alternative x is chosen from choice set S . A preference relation \succsim is a complete and transitive binary relation on X . We denote the strict part of \succsim with \succ , and the indifference part of \succsim with \sim . Next, we define the preference-oriented Luce rule.

Definition 1 An RCF ρ is a *preference-oriented Luce rule* (POLR) if there is a prefer-

⁷[McFadden et al. \(1973\)](#) argues that zero probability is empirically indistinguishable from a positive but small probability.

⁸[Hortacsu and Joo \(2016\)](#) illustrate this by using Dominick’s supermarket cola sales scanner data, which covers 100 chain stores in the Chicago area for 400 weeks, from September 1989 to May 1997.

ence relation \succsim on X and a weight function $v : X \rightarrow R_{++}$ such that for each choice set $S \in \Omega$,

$$\rho_x(S) = \begin{cases} \frac{v(x)}{\sum_{y \in \max(S, \succsim)} v(y)} & \text{if } x \in \max(S, \succsim) \\ 0 & \text{otherwise.} \end{cases}$$

In words, an agent first shortlists the \succsim -best alternatives from a choice set S , then chooses each shortlisted alternative with a probability that equals the alternative's relative weight in the shortlist. If no alternative is strictly better than another, that is if \succ is empty, then we obtain the classical Luce rule.

3 Results

Let ρ be an RCF. For each $S \in \Omega$ and $x \in S$, the **rejection likelihood of x in S** , denoted by $r(x, S)$, is the ratio of the probability that x is not chosen from S to the probability that x is chosen from S , i.e. $r(x, S) = \frac{1 - \rho_x(S)}{\rho_x(S)}$ if $\rho_x(S) > 0$, and $r(x, S) = \infty$ if $\rho_x(S) = 0$. Next, we introduce our new axiom.

Rejection supermodularity: For each $S, T \in \Omega$ and $x \in X$ such that $S \cap T = \{x\}$,

$$r(x, S) + r(x, T) \leq r(x, S \cup T).$$

It is easy to note that *rejection supermodularity* is equivalent to require that for each $x \in X$, $r(x, \cdot)$ is supermodular, i.e. for each $S, T \in \Omega$ such that $x \in S \cap T$, $r(x, S) + r(x, T) \leq r(x, S \cup T) + r(x, S \cap T)$. As we discuss in the introduction, *rejection supermodularity* strengthens the *regularity* axiom. Next, we present our characterization of preference-oriented Luce rule. We present the proof in Section 5.

Theorem 1 *An RCF ρ is a preference-oriented Luce rule if and only if ρ satisfies rejection supermodularity.*

In the representation, the preference relation is identified uniquely, and the identified weight function is unique up to multiplication by a positive scalar for each

equivalence class of the preference relation. In their empirical model, [Hortacsu and Joo \(2016\)](#) consider a consumer who assigns each product a (Luce) weight in a specific way. In the vein of [Helpman et al. \(2008\)](#), they model the consumer's choice as a two-stage procedure, in which the consumer first eliminates the products with weights lower than a given threshold, and then decides how much to consume of each remaining product according to its relative weight. The Luce weights are used both to determine which products are consumed with positive probability and their market share. Given an POLR, then we can rescale the weight function and create threshold values for the choice sets so that POLR representation fits into [Hortacsu and Joo \(2016\)](#)'s empirical model.⁹

The main difference between a preference-oriented Luce rule and the classic Luce rule is *positivity*, which requires each alternative be chosen from each choice set with positive probability, i.e. for each $S \in \Omega$ and $x \in S$, $\rho_x(S) > 0$. As a corollary to [Theorem 1](#), we obtain a new characterization of the Luce rule in terms of *positivity* and *rejection supermodularity*.

Corollary 1 *An RCF ρ is a Luce rule if and only if ρ satisfies positivity and rejection supermodularity.*

[Luce \(1959\)](#) characterizes his model in terms of *positivity* and IIA, which says that the ratio of choice probabilities between two alternatives does not depend on other alternatives in the choice set, i.e. for each $S \in \Omega$ and $x, y \in S$, $\rho_x(\{x, y\})/\rho_y(\{x, y\}) = \rho_x(S)/\rho_y(S)$. Luce also shows that under *positivity*, IIA is equivalent to the *choice axiom*, which requires that the probability of choosing an alternative x from a choice set S equals the probability of choosing any subset T of S , which contains x , multiplied by the probability of choosing x from T , i.e. for each $S, T \in \Omega$ with $T \subset S$ and $x \in T$, $\rho_x(S) = \sum_{y \in T} \rho_y(S) \cdot \rho_x(T)$.

⁹ In [Hortacsu and Joo \(2016\)](#)'s formulation, the threshold value is fixed, yet the weight of an alternative has a choice set dependent component. In mapping a POLR to a threshold model, we keep the weights fixed, but allow the threshold values depend on the choice sets such that as new alternatives are added to a choice set the threshold value does not decrease.

The behavioral message carried by *rejection supermodularity* is substantially different from that of IIA and the *choice axiom*. In that, *rejection supermodularity* strengthens *regularity*, whereas both IIA and the *choice axiom* indicate that agents chooses as if taking conditional probabilities with respect an underlying subjective probability measure.¹⁰This fundamental difference reflects itself as an additive formulation in *rejection supermodularity*, whereas IIA and the *choice axiom* are in multiplicative form. In our Lemma 5 and Lemma 6, we establish the indirect connection between *rejection supermodularity* and Luce’s axioms. The observation in Lemma 5 might be of mathematical interest for its own sake. This observation can be generalized to any collection of positive real numbers by using Cauchy–Bunyakovsky–Schwarz inequality.

4 Relation to the literature and final comments

We obtain a new axiomatic characterization of the Luce rule in terms of *positivity* and *rejection supermodularity* that substitutes Luce’s IIA and the *choice axiom*. To best of our knowledge this is the first alternative characterization of the Luce rule in its classical setting. There are two recent papers that offer new axiomatizations of the Luce rule in more structured choice environments. Gul et al. (2014) offer a characterization of the Luce rule for the “rich” choice environment. An implication of *richness* is that the alternative set should be infinite. Ahn et al. (2018) characterize the counterpart of the Luce rule in “average choice” environment, in which the alternative set is also infinite and endowed with a vector space structure that allows for the existence of convex combinations and averages. Here, we retain the original setting of Luce (1959), and consider a finite set of alternatives. These different assumptions on the choice environments lead to substantially different characterizations that shed light on different aspects of the Luce rule.

We show that *rejection supermodularity*, without *positivity*, characterizes POLR, which offers a structured extension of the Luce rule that accommodates zero proba-

¹⁰For the details see Lemma 2 of Luce (1959).

bility choices. [Hortacsu and Joo \(2016\)](#) develop an empirical choice model to accommodate zero choice probabilities. As we discuss in the previous section, by rescaling the Luce weights and choosing the thresholds properly, our POLR can also be represented in this form. Thus, besides its normative appeal, our characterization of POLR paves the way for identification of a model that has already received attention in the empirical literature.

There is a recent literature, which offers extensions of the Luce rule that accommodate zero choice probabilities. [Cerreia-Vioglio et al. \(2017\)](#) show that *the choice axiom* of Luce also characterizes POLR. We compare and contrast *rejection supermodularity* and the *choice axiom* in the previous section. Among others, [Yildiz \(2013\)](#) analyzes a specific POLR, in which the preference to be maximized is a weak order such that an alternative can be indifferent with at most one other alternative. He shows that this model is characterized by a random counterpart of [Plott \(1973\)](#)'s *path independence*. POLR generalizes this model by allowing an agent to be indifferent among any number of alternatives. [Echenique and Saito \(2015\)](#) and [Ahumada and Ulku \(2017\)](#) analyze a general Luce model that accommodates zero probability choices. In this model, an agent first forms his consideration sets in any arbitrary way, then applies the Luce rule. Our POLR can be seen as a general Luce rule with specific consideration set structure. A key difference between our work and these studies is our axiomatic characterization. In that, *rejection supermodularity* plays the main role in our results, whereas *cyclical independence*, which is a multiplicative extension of IIA, plays the main role in theirs.

5 Proof of Theorem 1

For any given RCF ρ , define the binary relation \succsim_ρ on X such that for each $x, y \in X$, $x \succsim_\rho y$ if and only if $\rho_x(\{x, y\}) > 0$. It follows that $x \succ_\rho y$ if and only if $\rho_x(\{x, y\}) = 1$. In the sufficiency part of the proof, we show that the \succsim_ρ relation is the relation that is to be maximized. In Lemma 3, we show that if an RCF ρ satisfies *rejection supermodularity*, then \succsim_ρ is complete and transitive. To prove this result, first we prove

the following two lemmas. Lemma 1 shows that if ρ satisfies *rejection supermodularity*, then ρ satisfies *regularity*, i.e. for each $S, T \in \Omega$ such that $S \subset T$, and $x \in S$, $\rho_x(T) \leq \rho_x(S)$.

Lemma 1 *If an RCF ρ satisfies rejection supermodularity, then ρ satisfies regularity.*

Proof. Let $T \in \Omega$ and $x \in T$. First, suppose $\rho_x(T) > 0$. Since for each $S \subset T$ such that $x \in S$, $r(x, S) + r(x, (T \setminus S) \cup \{x\}) \leq r(x, T)$, we have $r(x, S) \leq r(x, T)$. It follows that $\rho_x(T) \leq \rho_x(S)$. If $\rho_x(T) = 0$, then the conclusion follows directly. ■

Lemma 2 *Let ρ be an RCF that satisfies rejection supermodularity. For each $S \in \Omega$ and $x, y \in S$, if $\rho_x(S) > 0$ and $\rho_y(S) = 0$, then $\rho_x(\{x, y\}) = 1$.*

Proof. For each $S \in \Omega$ and $x, y \in S$, since ρ satisfies *rejection supermodularity*, we have $r(x, \{x, y\}) + r(x, S \setminus \{y\}) \leq r(x, S)$. Suppose $\rho_x(S) > 0$ and $\rho_y(S) = 0$, we show that $\rho_x(\{x, y\}) = 1$. We first show that $\rho_x(S \setminus \{y\}) = \rho_x(S)$. To see this, note that since ρ satisfies *regularity* by Lemma 1, for each $z \in S \setminus \{y\}$, $\rho_z(S \setminus \{y\}) \geq \rho_z(S)$. Suppose $\rho_x(S \setminus \{y\}) > \rho_x(S)$. Since the choice probabilities sum to one and $\rho_y(S) = 0$, there exists $z \in S \setminus \{x, y\}$ with $\rho_z(S \setminus \{y\}) < \rho_z(S)$. This is a contradiction. Now, since $\rho_x(S \setminus \{y\}) = \rho_x(S)$, we have $r(x, S \setminus \{y\}) = r(x, S)$. Therefore, $r(x, \{x, y\}) = 0$, which implies $\rho_x(\{x, y\}) = 1$. ■

Lemma 3 *If an RCF ρ satisfies rejection supermodularity, then \succsim_ρ is complete and transitive.*

Proof. Since either $\rho_x(\{x, y\}) \geq 0$ or $\rho_y(\{x, y\}) \geq 0$, \succsim_ρ is complete. To see that \succsim_ρ is transitive, by contradiction suppose there exist $x, y, z \in X$ such that $x \succsim_\rho y \succsim_\rho z$ and $z \succ_\rho x$, which means $\rho_x(\{x, z\}) = 0$. Now, consider $S = \{x, y, z\}$, since ρ satisfies *regularity* and $\rho_x(\{x, z\}) = 0$, $\rho_x(S) = 0$. Since $\rho_x(S) = 0$ and $x \succsim_\rho y$, it follows from Lemma 2 that $\rho_y(S) = 0$. Therefore, $\rho_z(S) = 1$. But, since ρ satisfies *regularity*, we must have $\rho_z(\{y, z\}) = 1$, which contradicts $y \succsim_\rho z$. ■

Lemma 4 *Let ρ be an RCF that satisfies rejection supermodularity. For each $S \in \Omega$ and $x \in S$, $\rho_x(S) > 0$ if and only if $x \in \max(S, \succsim_\rho)$.*

Proof. If part: For each $S \in \Omega$ and $x \in S$, $x \in \max(S, \succ_\rho)$ implies for each $y \in S \setminus \{x\}$, $\rho_x(\{x, y\}) > 0$. Since $\rho_y(S) > 0$ for some $y \in S$, it follows from Lemma 2 that $\rho_x(S) > 0$. Only if part: Suppose $\rho_x(S) > 0$. We show that $x \in \max(S, \succ_\rho)$. By contradiction, suppose there exists $y \in S$ such that $y \succ_\rho x$. It follows that $\rho_x(\{x, y\}) = 0$. Since ρ satisfies *regularity*, this implies that $\rho_x(S) = 0$, which contradicts $\rho_x(S) > 0$. ■

Next we present two critical results for the construction of the Luce weights.

Lemma 5 For each $x, y, z \in R_{++}$, $\frac{x}{1+x+xy} + \frac{y}{1+y+yz} + \frac{z}{1+z+zx} \leq 1$. Moreover, equality holds if and only if $xyz = 1$.¹¹

Proof. If we substitute $\frac{1}{1+x+xy} + \frac{x}{1+x+xy} + \frac{xy}{1+x+xy}$ instead of 1, then we get the following inequality

$$\frac{x}{1+x+xy} + \frac{y}{1+y+yz} + \frac{z}{1+z+zx} \stackrel{?}{\leq} \frac{1}{1+x+xy} + \frac{x}{1+x+xy} + \frac{xy}{1+x+xy}.$$

By rearranging the terms we get

$$\left[\frac{y}{1+y+yz} - \frac{xy}{1+x+xy} \right] + \left[\frac{z}{1+z+zx} - \frac{1}{1+x+xy} \right] \stackrel{?}{\leq} 0.$$

By equating denominators we get

$$\frac{y + xy + xy^2 - xy - xy^2 - xy^2z}{(1+y+yz)(1+x+xy)} + \frac{z + zx + xyz - 1 - z - zx}{(1+z+zx)(1+x+xy)} \stackrel{?}{\leq} 0.$$

Then,

$$\frac{y(1-xyz)}{(1+y+yz)(1+x+xy)} - \frac{1-xyz}{(1+z+zx)(1+x+xy)} \stackrel{?}{\leq} 0.$$

Finally we obtain

$$\frac{1-xyz}{1+x+xy} \left(\frac{xyz-1}{(1+y+yz)(1+z+zx)} \right) = \frac{-(1-xyz)^2}{(1+x+xy)(1+y+yz)(1+z+zx)} \stackrel{?}{\leq} 0.$$

It is clear that this inequality holds, and holds as an equality if and only if $xyz = 1$. ■

Lemma 6 Let ρ be an RCF that satisfies rejection supermodularity. For each distinct $x, y, z \in X$, if $\rho_x(x, y) \in (0, 1)$, $\rho_x(x, z) \in (0, 1)$, and $\rho_y(y, z) \in (0, 1)$, then $\frac{r(x, \{x, z\})}{r(x, \{x, y\})} = \frac{\rho_z(\{y, z\})}{\rho_y(\{y, z\})}$.

¹¹Selim Bahadir from the Mathematics department of Yildirim Beyazit University, Ankara, Turkey proved the extension of this result for more than three positive numbers.

Proof. First, note that for each $S, T \in \Omega$ with $S \cap T = \{x\}$, we have $\rho_x(S \cup T) = \frac{1}{1 + r(x, S \cup T)}$. Second, note that *rejection supermodularity* implies $\rho_x(S \cup T) \leq \frac{1}{1 + r(x, S) + r(x, T)}$. Now, consider the choice set $\{x, y, z\}$. It follows from these two observations that

$$\begin{aligned}\rho_x(x, y, z) &\leq \frac{1}{r(x, \{x, z\}) + 1 + r(x, \{x, y\})} = \frac{r(z, \{x, z\})}{1 + r(z, \{x, z\}) + r(z, \{x, z\})r(x, \{x, y\})}, \\ \rho_y(x, y, z) &\leq \frac{1}{r(y, \{x, y\}) + 1 + r(y, \{y, z\})} = \frac{r(x, \{x, y\})}{1 + r(x, \{x, y\}) + r(x, \{x, y\})r(y, \{y, z\})}, \\ \rho_z(x, y, z) &\leq \frac{1}{r(z, \{y, z\}) + 1 + r(z, \{x, z\})} = \frac{r(y, \{y, z\})}{1 + r(y, \{y, z\}) + r(y, \{y, z\})r(z, \{x, z\})}.\end{aligned}$$

Note that to obtain the rightmost terms, we multiply both the numerator and the denominator of the middle terms with $r(z, \{x, z\})$, $r(x, \{x, y\})$, and $r(y, \{y, z\})$ respectively. Now, consider the sum of the three rightmost terms. It follows from Lemma 5 that this sum is less than or equal to 1. On the other hand, since the sum of the three leftmost terms equals 1, the sum of the rightmost terms must be equal to 1. By Lemma 5, we know that this equality holds if and only if $r(x, \{x, y\}) \cdot r(y, \{y, z\}) \cdot r(z, \{x, z\}) = 1$. If we substitute $\frac{1}{r(x, \{x, z\})}$ instead of $r(z, \{x, z\})$, then we get $r(y, \{y, z\}) = \frac{r(x, \{x, z\})}{r(x, \{x, y\})}$. Since $r(y, \{y, z\}) = \frac{\rho_z(\{y, z\})}{\rho_y(\{y, z\})}$, we get the desired conclusion. ■

Now, we proceed to prove Theorem 1.

Only if part: Let ρ be a preference-oriented Luce rule represented by \succsim and v . First, suppose $r(x, S \cup T) = \infty$ for some $S, T \in \Omega$ such that $S \cap T = \{x\}$. It follows that $\rho_x(S \cup T) = 0$. This implies there exists $y \in S \cup T$ such that $y \succ x$. It follows that either $y \in S$ or $y \in T$. Therefore, either $r(x, S) = \infty$ or $r(x, T) = \infty$. Thus, we obtain $r(x, S) + r(x, T) = r(x, S \cup T)$. Next, suppose $r(x, S \cup T) < \infty$. It follows that $\rho_x(S \cup T) > 0$ and $x \in \max(S \cup T, \succsim)$. Let $S' = \max(S, \succsim) \setminus \{x\}$ and $T' = \max(T, \succsim) \setminus \{x\}$. Since $x \in \max(S, \succsim)$ and $x \in \max(T, \succsim)$, all the alternatives in S' and T' belong to the same indifference class of \succsim . Therefore, $\max(S \cup T, \succsim) = S' \cup T' \cup \{x\}$.

Now, since ρ is a preference-oriented Luce rule, we have

$$\rho_x(S) = \frac{v(x)}{v(x) + \sum_{y \in S'} v(y)}, \quad \rho_x(T) = \frac{v(x)}{v(x) + \sum_{y \in T'} v(y)}, \quad \text{and} \quad \rho_x(S \cup T) = \frac{v(x)}{v(x) + \sum_{y \in S' \cup T'} v(y)}$$

It follows that

$$r(x, S) = \frac{\sum_{y \in S'} v(y)}{v(x)}, \quad r(x, T) = \frac{\sum_{y \in T'} v(y)}{v(x)}, \quad \text{and} \quad r(x, S \cup T) = \frac{\sum_{y \in S' \cup T'} v(y)}{v(x)}$$

Since $S \cap T = \{x\}$, we have $S' \cap T' = \emptyset$. Thus, we obtain that $r(x, S) + r(x, T) = r(x, S \cup T)$.

If part: Suppose ρ is an RCF that satisfies *rejection supermodularity*. We construct a weak order \succsim and a weight function $v : X \rightarrow R_+$ that recovers the choices of ρ . We choose \succsim as the \succsim_ρ that is defined above. It follows from Lemma 3 that \succsim_ρ is complete and transitive. It follows from Lemma 4 that for each $S \in \Omega$ and $x \in S$, $\rho_x(S) > 0$ if and only if $x \in \max(S, \succsim_\rho)$.

Next, we construct the weight function $v : X \rightarrow R_{++}$. First for each $x \in X$, define $x_\sim = \{z \in X \mid x \sim_\rho z\}$. Next, we specify the weights for each of these equivalence classes. For each $x \in X$, if $x_\sim = \{x\}$, i.e. there is no $z \in X \setminus \{x\}$ with $z \sim_\rho x$, then let $v(x) = 1$. If there is a single $z \in X \setminus \{x\}$ with $z \sim_\rho x$, then let $v(x) = \rho_x(\{x, z\})$ and $v(z) = \rho_z(\{x, z\})$. If $|x_\sim| > 2$, then define $v(x) = 1$ for some fixed $x \in x_\sim$, and for each $z \in x_\sim \setminus \{x\}$, define $v(z) = r(x, \{x, z\})$. Thus we complete the construction of the weight function v .

Now, let $\hat{\rho}$ be the preference-oriented Luce rule defined by \succsim_ρ and v . First, we show that for each $x, y \in X$ we have $\rho_x(\{x, y\}) = \hat{\rho}_x(\{x, y\})$. To see this, note that for each $x, y \in X$, if $\rho_x(\{x, y\}) = 1$ ($\rho_x(\{x, y\}) = 0$), then $x \succ_\rho y$ ($y \succ_\rho x$). Therefore, $\hat{\rho}_x(\{x, y\}) = 1$ ($\hat{\rho}_x(\{x, y\}) = 0$). Next we show that if $x \sim_\rho y$, then $\frac{v(x)}{v(y)} = \frac{\rho_x(\{x, y\})}{\rho_y(\{x, y\})}$. Once we show this, then it directly follows that $\rho_x(\{x, y\}) = \hat{\rho}_x(\{x, y\})$. To see this, let $x^* \in x_\sim$ such that $v(x^*) = 1$ and for each $z \in x_\sim$, $v(z) = r(x^*, \{x^*, z\})$. If $x = x^*$ or $y = x^*$, then it directly follows from the definition of rejection likelihood that $\frac{v(x)}{v(y)} =$

$\frac{\rho_x(\{x,y\})}{\rho_y(\{x,y\})}$. If $x^* \notin \{x, y\}$, then it follows from Lemma 6 that $\frac{r(x^*, \{x^*, x\})}{r(x^*, \{x^*, y\})} = \frac{\rho_x(\{x, y\})}{\rho_y(\{x, y\})}$. Since $v(x) = r(x^*, \{x^*, x\})$ and $v(y) = r(x^*, \{x^*, y\})$, we obtain the desired conclusion.

Next, consider each $S \in \Omega$ and $x \in S$, note that by applying *rejection supermodularity* recursively, we get $r(x, S) \geq \sum_{y \in S \setminus \{x\}} r(x, \{x, y\})$. Next, we argue that this inequality should be satisfied as an equality. By contradiction, suppose there exist $S \in \Omega$ and $x^* \in S$ such that $r(x^*, S) > \sum_{y \in S \setminus \{x^*\}} r(x^*, \{x^*, y\})$. It follows from the previous step of the current proof that there exist Luce weights $\{v(x)\}_{x \in X}$ such that for each $x, y \in X$, $r(x, \{x, y\}) = v(y)/v(x)$. Next, multiply these weights with $1/\sum_{x \in S} v(x)$. Let v' denote the new Luce weights. Now, note that for each $x, y \in X$, $r(x, \{x, y\}) = v'(y)/v'(x)$. Moreover, since $\sum_{x \in S} v'(x) = 1$, for each $x \in S$, $\sum_{y \in S \setminus \{x\}} r(x, \{x, y\}) = \frac{1-v'(x)}{v'(x)}$. On the other hand, by definition, $r(x, S) = (1-\rho_x(S))/\rho_x(S)$. It follows from our initial supposition that for each $x \in S$, $\rho_x(S) \geq v'(x)$ and $\rho_{x^*}(S) > v'(x^*)$. But, this implies $\sum_{x \in S} \rho_x(S) > \sum_{x \in S} v'(x) = 1$. Thus, we get a contradiction. Finally, since for each $S \in \Omega$ and $x \in S$, $r(x, S) = \sum_{y \in S \setminus \{x\}} r(x, \{x, y\})$, choices from pairs of alternatives uniquely specify ρ . Since for each pair of alternatives ρ and $\hat{\rho}$ assign the same choice probabilities, we get $\rho = \hat{\rho}$. It follows that ρ is a preference-oriented Luce rule.

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