Equitable stable matchings under modular assessment

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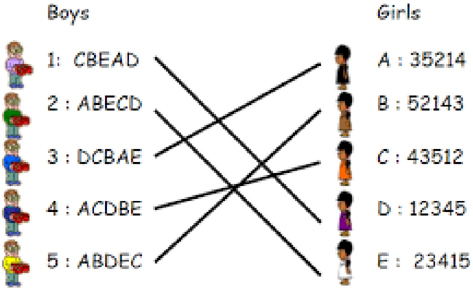
joint with


stable matching lattice
a matching μ is stable if there is no man-woman pair who prefer each other to their mates at μ.
a matching μ is men-wise better than another matching μ', denoted by μ ⊳_M μ', if for each m ∈ M, μ(m) ≻_m μ'(m) or μ(m) = μ'(m).
Thm: The set of stable matchings S together with the relation ⊳_M forms a lattice: If μ and μ' are stable, then μ ∨ μ' and μ ∧ μ' are stable.

a two-sided matching (marriage) problem



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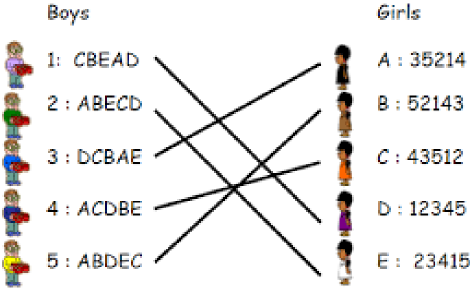

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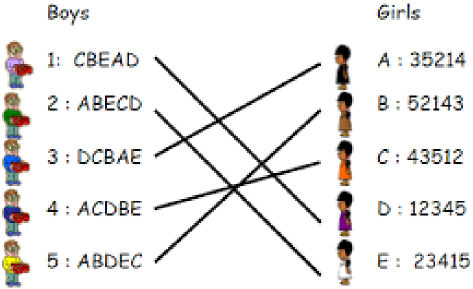

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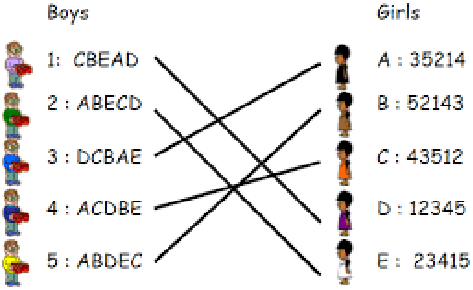

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a problem: “$\succ = \{\succ_i\}_{i \in N}$”

M \ W	a	b	c	d	e	f
1	1, 6	2, 5	3, 4	4, 3	5, 1	6, 2
2	2, 5	1, 6	6, 2	3, 4	4, 3	5, 1
3	5, 1	6, 2	1, 6	2, 5	3, 4	4, 3
4	4, 3	5, 1	2, 5	1, 6	6, 2	3, 4
5	3, 4	4, 3	5, 1	6, 2	1, 6	2, 5
6	6, 2	3, 4	4, 3	5, 1	2, 5	1, 6

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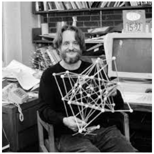

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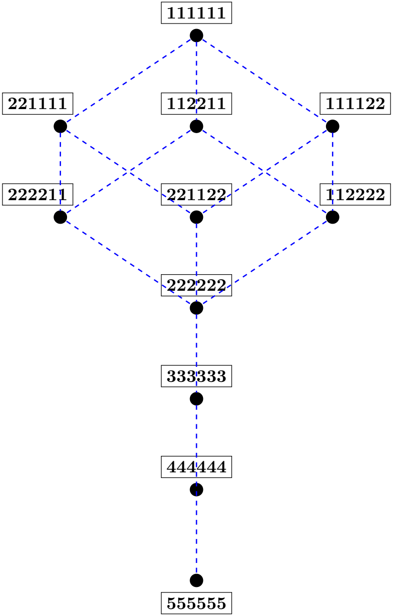

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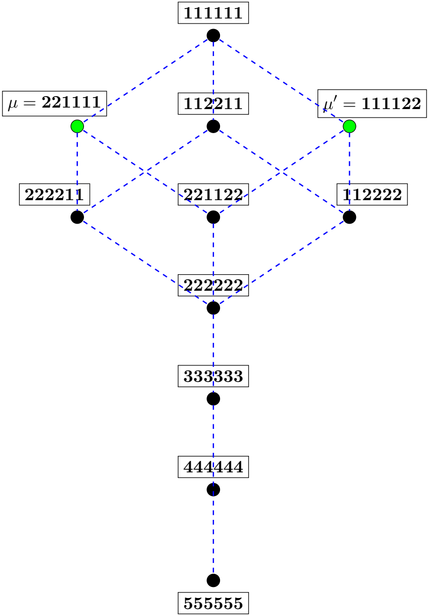

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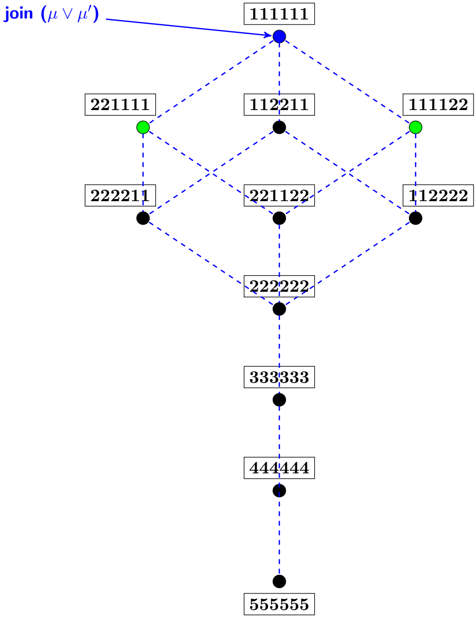

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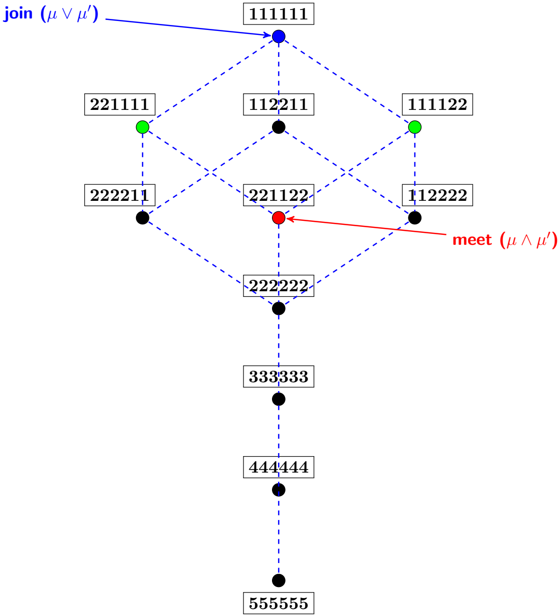

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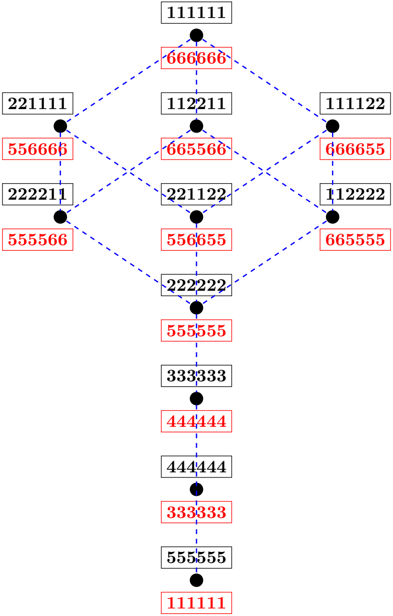
Thm: *The set of stable matchings $\mathcal{S}$ together with the relation $\rhd_M$ forms a lattice with the polarity property: for each $\mu, \mu' \in \mathcal{S}$,*

$$\mu \rhd_M \mu' \textit{ iff } \mu' \rhd_W \mu.$$

.

polarity


111111
666666
221111
556666
112211
665566
111122
666655
222211
555566
221122
556655
112222
665555
222222
555555
333333
444444
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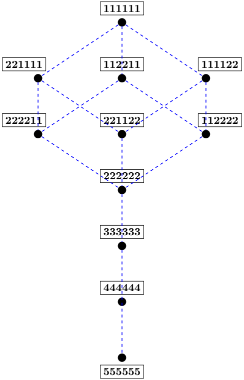

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- The breadth of possibilities calls for a "foundational framework" to address the issue of equity and social welfare.

Our aim: *To introduce such a framework together with a new class of solutions.*

outline

Part 1 The framework:

- Two basic axioms $\longrightarrow$ stable matching rules
- Modularity

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- Two basic axioms $\longrightarrow$ stable matching rules
- Modularity for analytical tractability along with clarity and richness.

Part 2 A new class of equity notions

The framework

stable matching rules

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- **Stability:** For each problem $\succ$ the chosen matchings $\pi(\succ)$ must be stable.
- **Invariance under stability:** If the set of stable matchings for two problems $\succ$ and $\succ'$ are the same, then the chosen matchings must be the same, i.e $\pi(\succ) = \pi(\succ')$.

Definition: A matching rule π is a stable matching rule if π satisfies *stability* and *invariance under stability*.

invariance under stability?

YES	*NO*
median stable matchings	egalitarian stable matchings
(*Teo & Sethuraman'98*)	(*McVitie & Wilson'71*)
medians of the lattice	minimum regret matchings
(*Cheng'10*)	(*Knuth'76*)
center stable matchings	sex-equal stable matchings
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$\succ_w$	$\succ_m$
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2	2
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- this sum is constant among all stable matchings, and therefore does not differentiate any stable matching from the others.

Modularity

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Definition: A stable matching rule π is modular if for each problem $\succ$, there exists a modular $F : \mathcal{S}(\succ) \to \mathbb{R}$ s.t. $\pi(\succ)$ is the set of matchings that minimize (optimize) F, that is

$$\pi(\succ) = \text{argmin}_{\mu \in \mathcal{S}(\succ)} F(\mu)$$

.

why modularity?

Proposition 1: F is modular if and only if for each $i \in N$, there exists $F_i : A_i \to \mathbb{R}$ s.t. for each $\mu \in \mathcal{S}(\succ)$,

$$F(\mu) = \sum_{i \in N} F_i(\mu(i))$$

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Clarity:

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Clarity: The social value of a stable matching is obtained by adding $F_i(\mu(i))$ for each agent i.

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Tractability: Minimizers of F are isomorphic to min cuts of a specific flow network (obtained from Picard'76).

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- A simple(r) test for verifying modularity?

an answer

Theorem 1: *A stable matching rule* π *is modular if and only if* π *satisfies convexity.*

an answer

Theorem 1: *A stable matching rule π is modular if and only if π satisfies convexity.*

Convexity: Stable "mixtures" of chosen matchings are also chosen.

Convexity: If $\mu', \mu'' \in \pi(\succ)$ and there is a stable matching μ s.t. $\mu(m) \in \{\mu'(m), \mu''(m)\}$ for every $m \in M$, then $\mu \in \pi(\succ)$.

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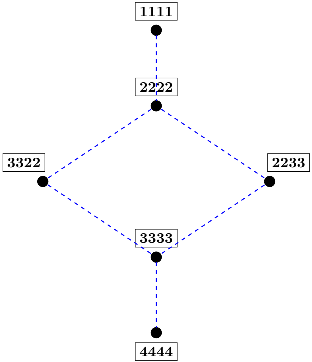

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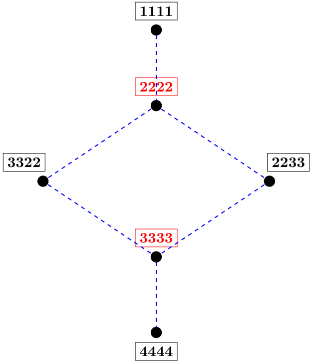

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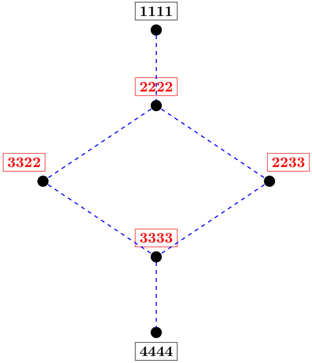

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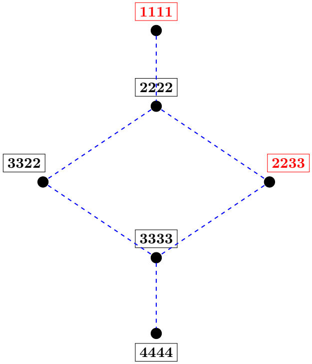

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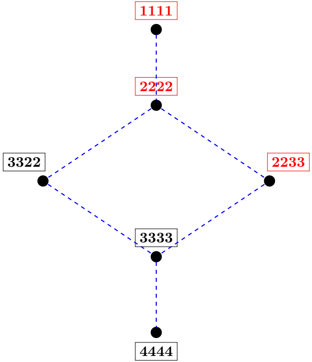

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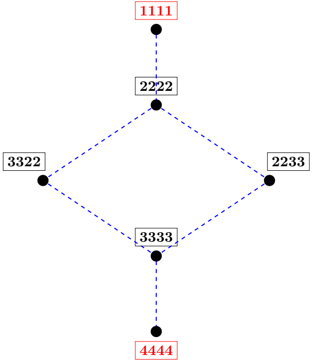

an answer

Theorem 1: *A stable matching rule* π *is modular iff* π *satisfies convexity.*

Proof

another answer

another answer

Theorem: *A stable matching rule π is modular iff π satisfies independence of irrelevant rankings.*

Part II

independence of irrelevant rankings

- $\pi_i(\succ)$ is the set of agents matched to i at some $\mu \in \pi(\succ)$.

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$\succ_i$
1
2
3
4
5

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$$\begin{array}{c} \succ_i \\ \hline 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

- Then, the π-transformed problem $\succ^\pi$ obtained from $\succ$ s.t. for each agent i, each member of $\pi_i(\succ)$ is moved to the top of i's preferences by preserving the relative rankings elsewhere.

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$$\begin{array}{ccc} \underline{\succ_i} & & \underline{\succ_i^\pi} \\ 1 & & 3 \\ 2 & & 5 \\ 3 & \longrightarrow & 1 \\ 4 & & 2 \\ 5 & & 4 \end{array}$$

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independence of irrelevant rankings

$\succ_i$		$\succ_i^\pi$
1		3
2		5
3	$\longrightarrow$	1
4		2
5		4

independence of irrelevant rankings

$\succ_i$		$\succ_i^\pi$
1		3
2		5
3	$\longrightarrow$	1
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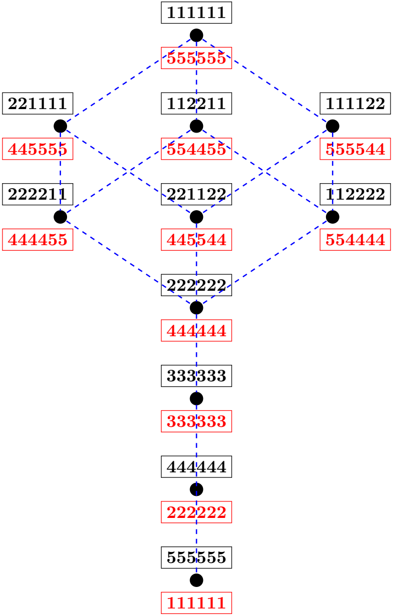
Independence of irrelevant rankings: If a matching that is stable in the original problem remains stable in the transformed problem, then it must be chosen in the initial problem.

Thm 1's relation to the literature

- Kreps'79 and Chambers & Echenique'09 provide representations for modular preferences over lattices under the additional assumption of *monotonicity*.
- A stable matching rule that satisfy *monotonicity* would choose one of the extremal matchings.

A new (class of) equity notion(s)

which matching seems most equitable?



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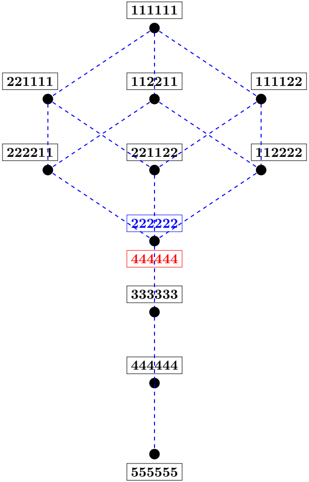

- the *median stable matching* (Teo & Sethuraman'98), its extension by (Cheng'10), and the *center* (CMS'16) is [222222] in which each man is matched to his second ranked woman.

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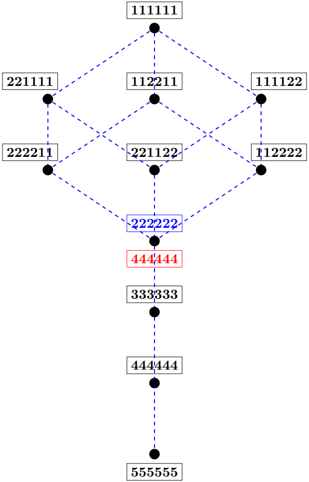

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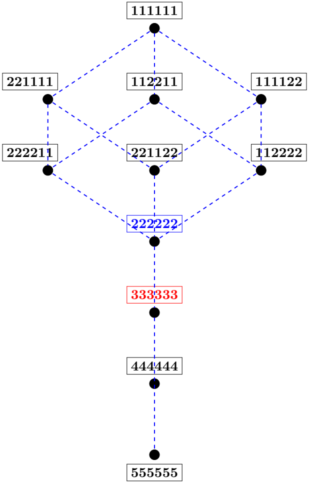

- However, the matching [333333] is equitable in the sense that each agent is matched to his/her median attainable mate.

median attainable mate

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1
2
3
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- Let med_i^A be the attainable mate for agent i with the lowest attainable median rank, i.e. $Rank_i^A(med_i^A) = \lfloor (|A_i| + 1)/2 \rfloor$.

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$\pi(\succ)$ *is the set of matchings that minimize:*

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$\pi(\succ)$ can be found in P-time since:

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2. This is a modular stable matching rule.

a new fairness notion

Equity undominance: *If a matching is chosen, then there is no other stable matching in which each agent's mate is same or closer to their median attainable mate* (med_i^A).

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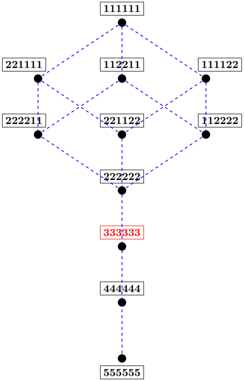
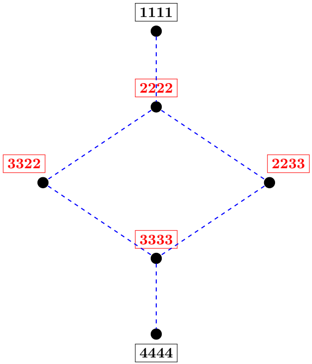
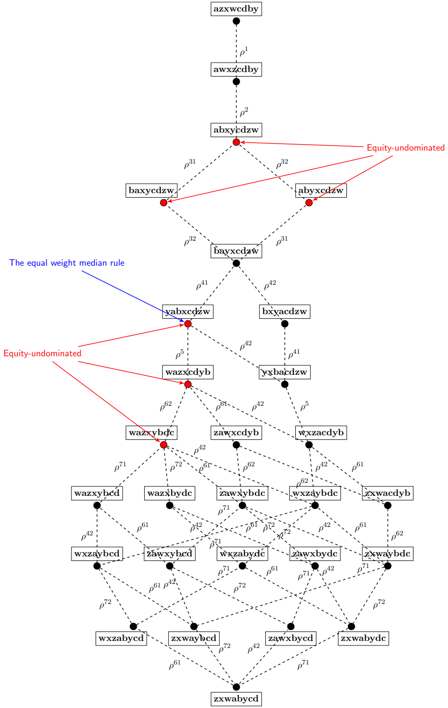


a new fairness notion

Equity undominance: *If $\mu \in \pi(\succ)$, then there is no $\mu' \in \mathcal{S}(\succ)$ s.t. $\mu'(i)$ is closer to med_i^A than $\mu(i)$ for every agent i with $\mu(i) \neq \mu'(i)$.*




azxwcdby
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abxycdzw
Equity-undominated
baxycdzw
abyxcdzw
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yabxcdzw
bxyacdzw
Equity-undominated
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yxbacdzw
wazxybdc
zawxcdyb
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wazxbydc
zawxybdc
wxzaybdc
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zawxybcd
wxzabydc
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result

Equity undominance: *If a matching is chosen, then there is no other stable matching in which each agent's mate is same or closer to their median attainable mate (med_i^A).*

Theorem 2: *Let π be a stable matching rule. Then, π satisfies convexity and equity undominance iff $\pi(\succ)$ is the set of matchings that maximize:*

$$\mu(i)$$

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Theorem 2: *A stable matching rule π satisfies convexity and equity undominance iff $\pi(\succ)$ is the set of matchings that maximize:*

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where $F_i : A_i \to \mathbb{R}$ is unimodal with mode med_i^A for each $i \in N$.

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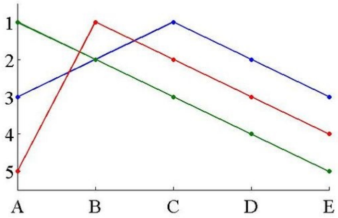

more generally

Equity undominance: *If a matching is chosen, then there is no other stable matching in which each agent's mate is same or closer to their ideal attainable mate ($I(i) \in A_i$).*

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more generally

Theorem 2: *A stable matching rule π satisfies convexity and equity undominance iff $\pi(\succ)$ is the set of matchings that maximize:*

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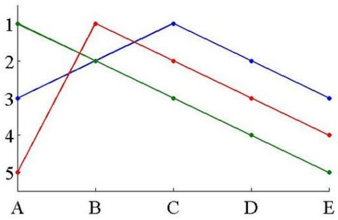

extensions

- As for the many-to-one and many-to-many matchings, the set of stable matchings forms a distributive lattice under familiar restrictions (Alkan'01, Alkan & Gale'03).

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Conc

extensions

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Conc

Proof sketch for

Theorem 1: *A stable matching rule π is modular iff π satisfies convexity.*

Conc

roadmap for the sketch

- convexity ⇒ modularity:
  1. Connection to the *rotations poset*.
  2. *Hyperrotations* and *constraints*.
  3. *Partition lemma* and construction of F.
- modularity ⇒ convexity

Step 1: Rotations

- Rotations are the incremental changes that transform a stable matching μ into another stable matching μ' s.t. $\mu \rhd_M \mu'$ and there is no other stable matching μ'' s.t. $\mu \rhd_M \mu'' \rhd_M \mu'$ (Irving'85).

$$\rho^{11} = [(m_1, w_1), (m_2, w_2)]$$
$$\rho^{12} = [(m_3, w_3), (m_4, w_4)]$$
$$\rho^{13} = [(m_5, w_5), (m_6, w_6)]$$
$$\rho^2 = [(m_1, w_2), (m_4, w_3), (m_5, w_6), (m_2, w_1), (m_3, w_4), (m_6, w_5)]$$
$$\rho^3 = [(m_1, w_3), (m_2, w_4), (m_3, w_5), (m_4, w_6), (m_5, w_1), (m_6, w_2)]$$
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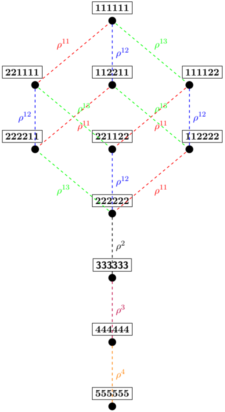
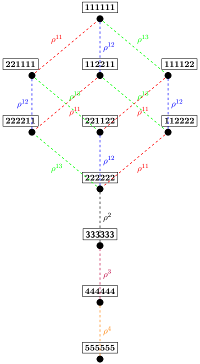
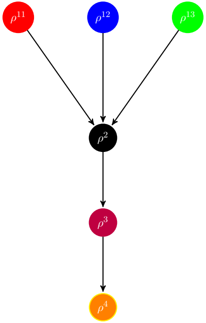


Step 2





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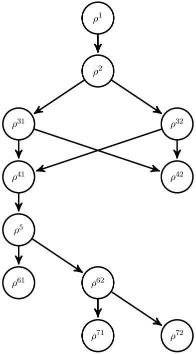
another rotation poset



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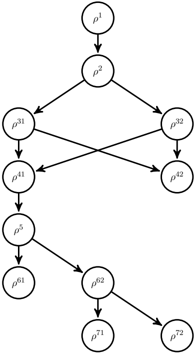

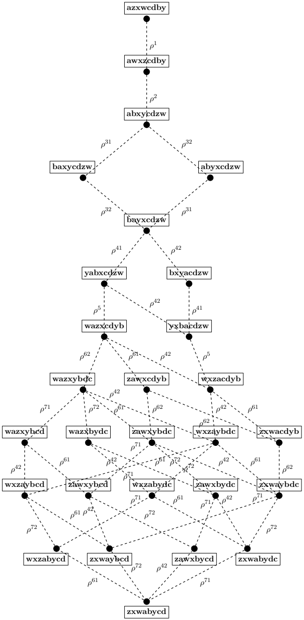

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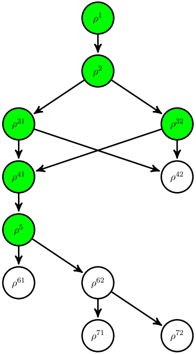

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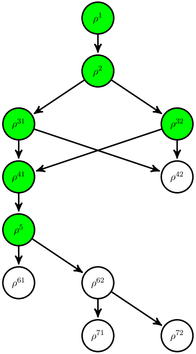
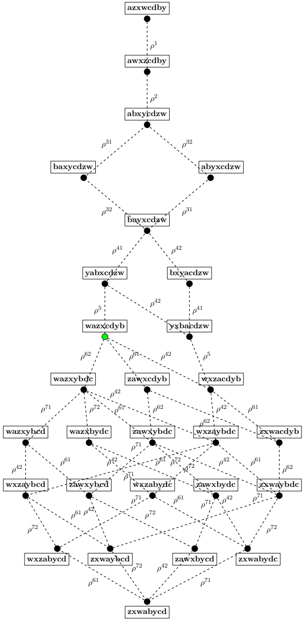




Step 1

It follows from Irving & Leather'86 that:

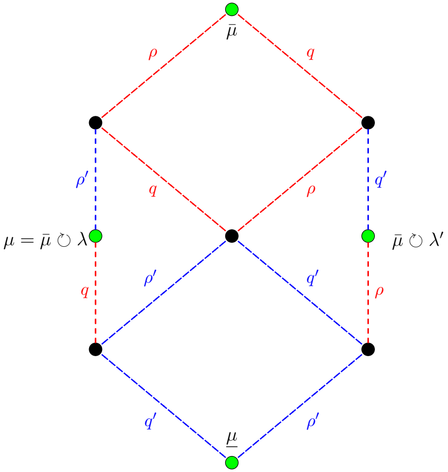
- Each attainable pair (m, w) is contained in a unique rotation.
- The closed sets of rotations with the set containment relation $\langle Cl(\mathcal{R}), \subset \rangle$ is a lattice that is *order isomorphic* to $\langle \mathcal{S}, \rhd_M \rangle$ (similar to Birkhoff''s Representation Theorem).

Thus, our problem boils down to assigning a weight $g(\rho)$ to each rotation ρ s.t.

$$\pi(\succ) = \mathrm{argmin}_{\mu \in \mathcal{S}} \sum_{\rho \in R_\mu} g(\rho)$$

.

Step 2: Hyper-rotations



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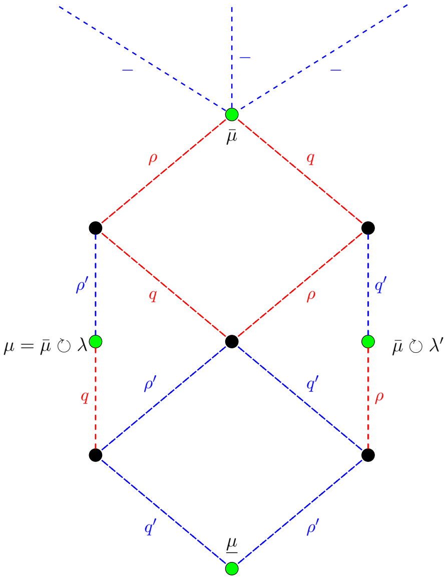

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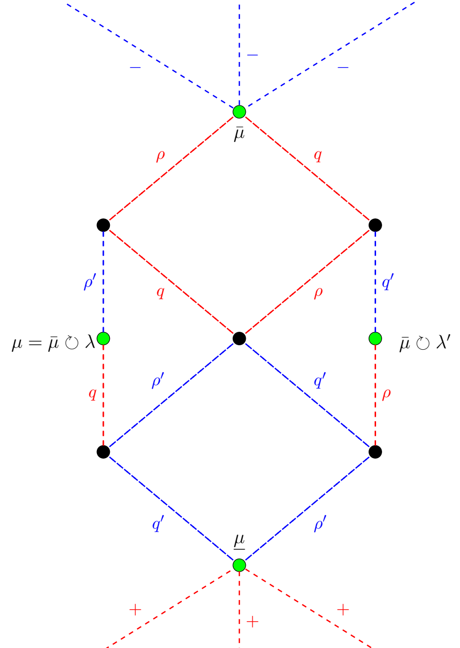

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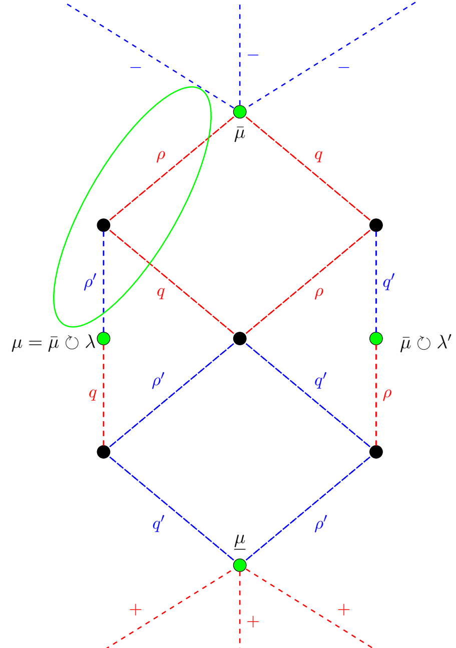

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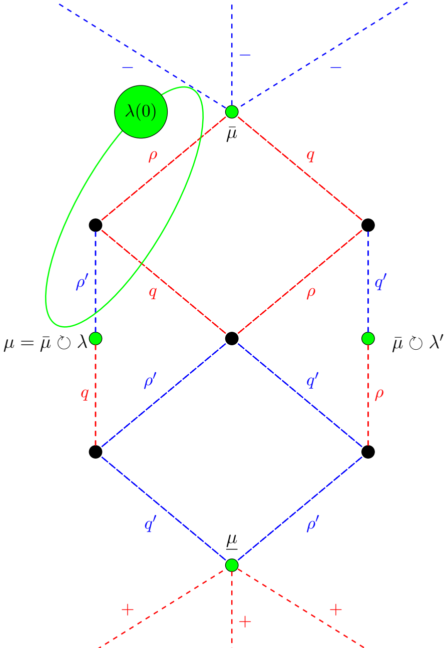

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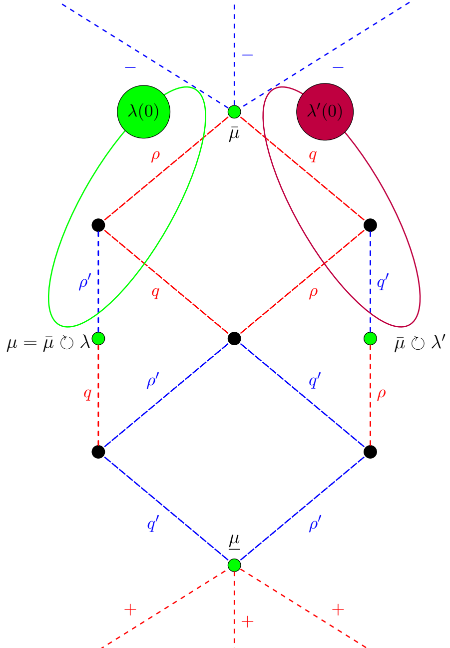

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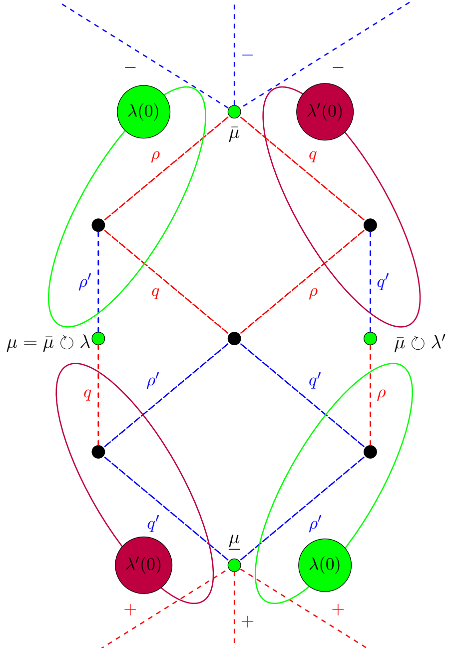
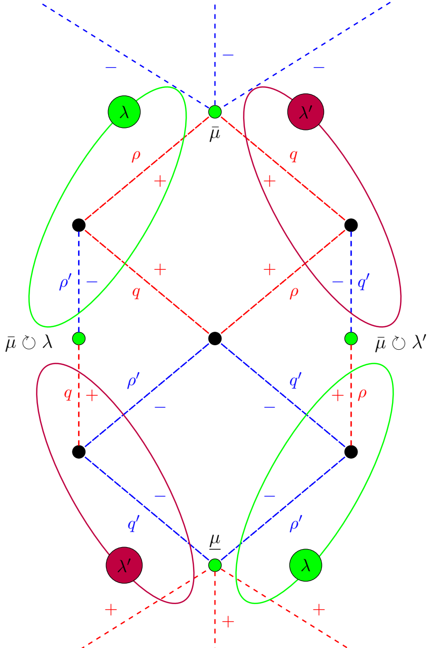


Step 2: Constraints



Step 2: Constraints

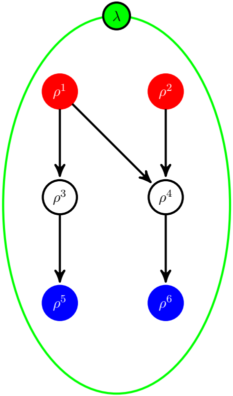
Thus, our problem boils down to designing the weight function g s.t. for each hyper-rotation λ,

- $\sum_{\rho\in\lambda} g(\rho) = 0$.
- for each $R \subsetneq \lambda$ that is (relatively) closed in λ,

$$\sum_{\rho\in R} g(\rho) > 0$$

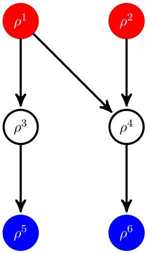
.

Step 3: Construction of g



- $\bar{\lambda} = \{q \in \lambda$ without any predecessor$\} = \{\rho^1, \rho^2\}$.
- $\underline{\lambda} = \{\rho \in \lambda$ without any successor$\} = \{\rho^5, \rho^6\}$.

construction of g: “preloading”



	$\lambda^{\downarrow}(\rho)$	$\lambda^{\uparrow}(\rho)$
ρ_1	$\{\rho_5, \rho_6\}$	$\{\rho_1\}$
ρ_2	$\{\rho_6\}$	$\{\rho_2\}$
ρ_3	$\{\rho_5\}$	$\{\rho_1\}$
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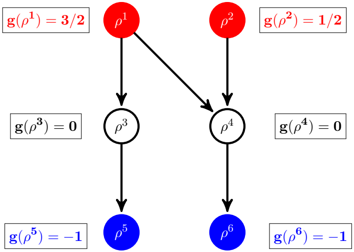
- $\lambda^{\uparrow}(\rho) = \{q \in \bar{\lambda} \mid q \to p\}$ & $\lambda^{\downarrow}(\rho) = \{q \in \underline{\lambda} \mid \rho \to q\}$.

construction of g: "preloading"



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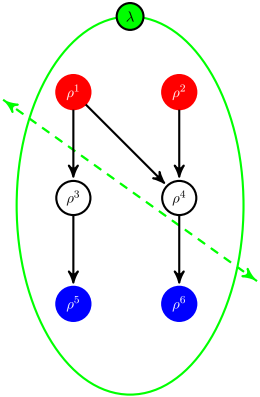
ρ_5	$\{\rho_5\}$	$\{\rho_1\}$
ρ_6	$\{\rho_6\}$	$\{\rho_1, \rho_2\}$

$$g(\rho) = \begin{cases} -1 & \text{if } \rho \in \underline{\lambda}, \\ \displaystyle\sum_{q \in \lambda^{\downarrow}(\rho)} \frac{1}{|\lambda^{\uparrow}(q)|} & \text{if } \rho \in \bar{\lambda}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

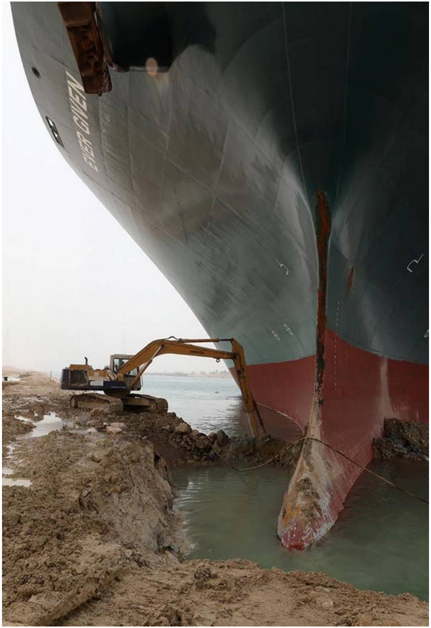
Step 3: Role of convexity

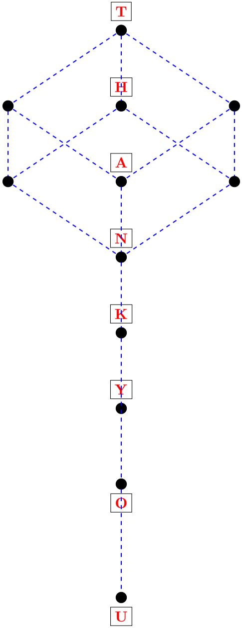
Bisection lemma: Each hyper-rotation λ is a connected subset of the rotations poset, i.e. for each bisection $\{P, P'\}$ of λ, there exist $\rho \in P$ and $\rho' \in P'$ s.t. $\rho \to \rho'$ or $\rho' \to \rho$.



conclusion

conclusion

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modularity $\Rightarrow$ convexity

- Modularity implies that $\pi(\succ)$ is a sublattice.

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- Since μ is a mixture, for each $\rho \in \lambda$ and each $\rho' \in \lambda'$, $\rho \cap \rho' = \emptyset$. Therefore ρ and ρ' are independent.

modularity ⇒ convexity

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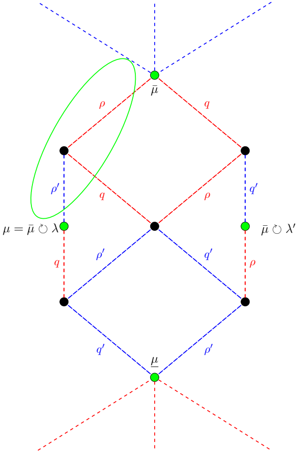
- Let $\lambda = R_\mu \setminus R_{\bar{\mu}}$ and $\lambda' = R_{\underline{\mu}} \setminus R_\mu$.
- Since λ and λ' are independent, $F(\bar{\mu} \circlearrowright \lambda') < F(\bar{\mu})$.



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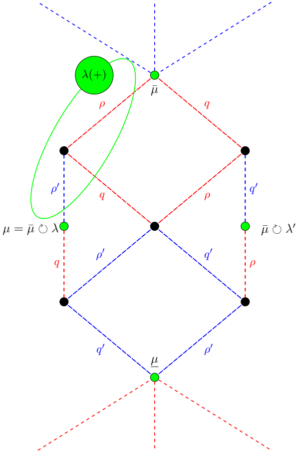

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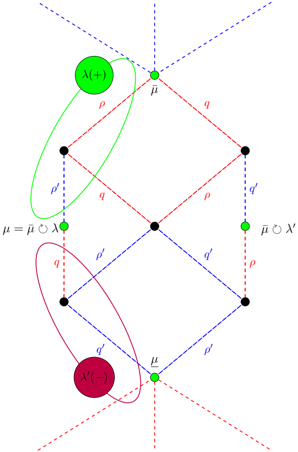

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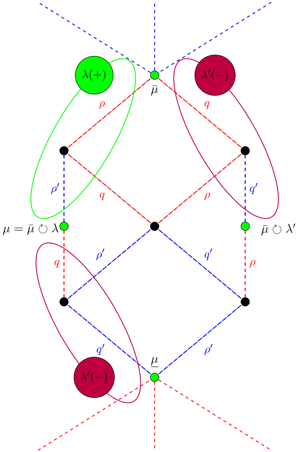

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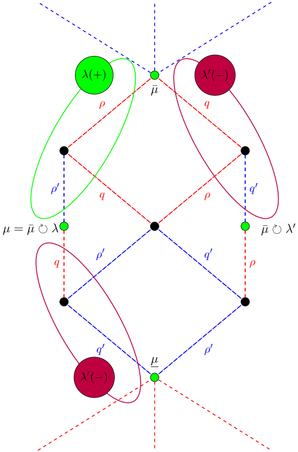

conclusion

Thank you!